Advanced Mathematical Methods for Economic Decisions

Foundations of continuous-time deterministic dynamic models

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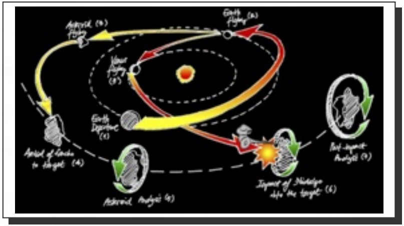
Course overview

Main topics and objectives



- Linear and nonlinear modeling
- Dynamical systems perspective
- Equilibria, stability, and local analysis

Given an *open* interval $I \subseteq \mathbb{R}$ and a *locally integrable* function $g : I \longrightarrow \mathbb{R}$, we consider

$$y'(t) = g(t), \quad t \in I,$$

where $y \equiv y_g : I \longrightarrow \mathbb{R}$ is a *differentiable* function on I and is *to be determined*.

Solution: for any $t_0 \in I$ and $c \in \mathbb{R}$,

$$y(t) \equiv y_{g,t_0,c}(t) = \int_{t_0}^{t} g(\tau)\, d\tau + c, \quad t \in I.$$

Its explicit computation depends on the analytical structure of g.

Uniqueness: an *initial condition*

$$y(t_0) = y_0,$$

with $y_0 \in \mathbb{R}$, selects one specific solution among them.

Meaning: the solution reconstructs the *state variable* y from its (*instantaneous*) *rate of change*, capturing the *cumulative effect* of the *forcing term* g.

Ordinary differential equations (ODEs) of order n

Definition and conceptual interpretation

Let $I \subseteq \mathbb{R}$ be an *open* interval and $n \in \mathbb{N} \setminus \{0\}$. Given $F\colon I \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, we consider

$$F\big(t, y(t), y'(t), \ldots, y^{(n)}(t)\big) = 0, \quad t \in I,$$

where

$$y \equiv y_F\colon I \longrightarrow \mathbb{R}$$

is an n-*times differentiable* function on I and is *the unknown*, and the equation is assumed to genuinely involve its n-th derivative.

Solution: in general, *no* explicit closed-form expression for y exists; *qualitative* and *numerical* analysis therefore prevail.

Meaning: the solution reconstructs the state from its *successive rates of change*.

Relevance: differential equations capture the *temporal structure of systems*—across physics, chemistry, biology, economics, finance, and engineering—providing a unified framework to analyze their *evolution*, *equilibria*, *stability*, and *qualitative asymptotic dynamics*.

Normal (or explicit) form of ODEs

General and first-order case

Given $f\colon I \times \mathbb{R}^n \longrightarrow \mathbb{R}$, we consider

$$y^{(n)}(t) = f\big(t, y(t), \dots, y^{(n-1)}(t)\big), \quad t \in I$$

(by convention, $y^{(0)} := y$).

First-order ($n = 1$):

$$y'(t) = f\big(t, y(t)\big), \quad t \in I.$$

Newton's notation: $\dot{y} := y'$ (and $\ddot{y} := y''$).

Geometric intuition: a *first*-order ODE in normal form defines a *vector field* as

$$(t, y) \mapsto \big(1, f(t, y)\big), \quad (t, y) \in I \times \mathbb{R},$$

prescribing the *local direction and velocity* at every state point; each solution $y(t)$, $t \in I$ corresponds to the *integral curve*

$$t \mapsto \big(t, y(t)\big)$$

of this field, and, ideally, such curves *neither split nor intersect*.

Autonomous ODEs in normal form

Continuous-time dynamical systems

Given $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, we consider

$$y^{(n)}(t) = f\big(y(t), \dots, y^{(n-1)}(t)\big), \quad t \in I.$$

Scalar continuous-*time* dynamical systems: when $n = 1$; that is,

$$y'(t) = f\big(y(t)\big), \quad t \in I.$$

This corresponds to an autonomous *first*-order ODE in normal form.

Remark: every n-th order ODE in normal form admits an *equivalent representation* as a *vector* continuous-time dynamical system of dimension N,

$$\boldsymbol{y}'(t) = \boldsymbol{f}\big(\boldsymbol{y}(t)\big), \quad t \in I,$$

for suitable $\boldsymbol{y}\colon I \longrightarrow \mathbb{R}^N$ and $\boldsymbol{f}\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$, where $N = n$ in the autonomous case and $N = n + 1$ otherwise (vector differentiation is taken componentwise).

Autonomous ODEs in normal form

Higher-order ODEs and their reduction to a first-order system

Low–order cases:

$$y'(t) = f\big(t, y(t)\big), \quad t \in I \quad \Longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} t \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ f\big(t, y(t)\big) \end{bmatrix}, \quad t \in I;$$

$$y''(t) = f\big(y(t), y'(t)\big), \quad t \in I \quad \Longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y'(t) \\ f\big(y(t), y'(t)\big) \end{bmatrix}, \quad t \in I;$$

$$y''(t) = f\big(t, y(t), y'(t)\big), \quad t \in I \quad \Longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} t \\ y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ y'(t) \\ f\big(t, y(t), y'(t)\big) \end{bmatrix}, \quad t \in I;$$

$$\vdots \qquad \vdots \qquad \vdots$$

Cauchy–Lipschitz Theorem

Local existence and uniqueness of solutions for first-order initial value problems

Theorem (Cauchy–Lipschitz/Picard–Lindelöf)

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *with* $\alpha < \beta$ *and* $\gamma < \delta$, *and let* $f : [\alpha, \beta] \times [\gamma, \delta] \to \mathbb{R}$ *be continuous on the product domain and locally Lipschitz continuous in the second variable. Then, for any* $(t_0, y_0) \in \,]\alpha, \beta[\times]\gamma, \delta[$, *there exists* $\varepsilon \equiv \varepsilon_{f,t_0,y_0} \in \,]0, \infty[$ *such that the initial value problem*

$$\begin{cases} y'(t) = f\big(t, y(t)\big) \\ y(t_0) = y_0 \end{cases}$$

admits a unique solution y *on the open interval* $]t_0 - \varepsilon, t_0 + \varepsilon[\,\cap\,]\alpha, \beta[$.

Local Lipschitzianity: there exist $L \equiv L_{f,t_0,y_0}, \eta \equiv \eta_{f,t_0,y_0} \in \,]0, \infty[$ such that, for all $t \in \,]t_0 - \eta, t_0 + \eta[$ and all $y_1, y_2 \in \,]y_0 - \eta, y_0 + \eta[$,

$$\big|f(t, y_2) - f(t, y_1)\big| \le L\big|y_2 - y_1\big|.$$

Examples: functions whose y*–partial derivative* is locally bounded.

Corollaries of the Cauchy–Lipschitz Theorem

Global existence of solutions for first-order initial value problems

Corollary

Let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on the product domain and locally Lipschitz continuous in the second variable. For $(t_0, y_0) \in I \times \mathbb{R}$, consider the initial value problem

$$\begin{cases} y'(t) = f\big(t, y(t)\big) \\ y(t_0) = y_0 \end{cases} .$$

Then any (local) solution y of this problem extends to the entire interval I, with no finite–time blow–up, in any of the following cases.

- *y remains bounded up to the endpoints of the domain I.*
- *f is globally Lipschitz continuous in the second variable.*
- *f satisfies a (sub)linear growth condition in the second variable.*

Corollaries of the Cauchy–Lipschitz Theorem

Non-intersection of solution trajectories for first-order ODEs in normal form

Corollary

Let $f: I \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous on the product domain and locally Lipschitz continuous in the second variable, and consider the first-order ODE

$$y'(t) = f\big(t, y(t)\big), \quad t \in I.$$

If $y, \tilde{y}: I \longrightarrow \mathbb{R}$ are two distinct solutions of this equation, then

$$\forall\, t \in I,\ y(t) \neq \tilde{y}(t).$$

Corollary

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be locally Lipschitz continuous, and consider the scalar continuous-time dynamical system

$$y'(t) = f\big(y(t)\big), \quad t \in I.$$

If $y: I \longrightarrow \mathbb{R}$ is not a steady-state (i.e., constant) solution of this equation, then

$$\forall\, t \in I,\ f\big(y(t)\big) \neq 0.$$

Basic examples: linear ODEs in normal form

General and low–order cases

Given $a_{n-1}, \ldots, a_1, a_0, b\colon I \longrightarrow \mathbb{R}$, we consider

$$y^{(n)}(t) = \sum_{k=0}^{n-1} a_k(t) y^{(k)}(t) + b(t), \quad t \in I.$$

Homogeneous: if, on I,

$$b \equiv 0.$$

With constant coefficients: when $a_{n-1}, \ldots, a_1, a_0$ are identically constant on I.

Autonomous: with constant coefficients and b constant on I.

First-order ($n = 1$):

$$y'(t) = a_0(t) y(t) + b(t), \quad t \in I.$$

Second-order ($n = 2$):

$$y''(t) = a_1(t) y'(t) + a_0(t) y(t) + b(t), \quad t \in I.$$

First-order linear ODEs

Superposition principle, solution, and autonomous case

Given two (*not* necessarily continuous) *locally integrable* functions $a, g : I \longrightarrow \mathbb{R}$, we consider

$$y'(t) = a(t)y(t) + g(t), \quad t \in I.$$

Superposition principle (valid for linear ODEs of *any* order): every solution y is the *sum* of a *particular* solution of the equation and a solution y_h of the associated *homogeneous* equation

$$y_h'(t) = a(t)y_h(t), \quad t \in I.$$

Solution: for any $t_0 \in I$ and $c \in \mathbb{R}$,

$$y(t) = e^{\int_{t_0}^{t} a(s)ds} \left[\int_{t_0}^{t} g(\tau) e^{-\int_{t_0}^{\tau} a(s)ds} d\tau + c \right], \quad t \in I.$$

Autonomous case (dynamical system): for $a, b \in \mathbb{R}$ with $a \neq 0$,

$$y'(t) = ay(t) + b, \quad t \in I \quad \Longleftrightarrow \quad y(t) = -\frac{b}{a} + ce^{at}, \quad t \in I,$$

for any $c \in \mathbb{R}$.

Nonlinear examples: first-order separable ODEs

On explicit solvability beyond linear models

Given $g\colon I \longrightarrow \mathbb{R}$ *locally integrable* and $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ *locally Lipschitz continuous*, we consider

$$y'(t) = g(t) f\big(y(t)\big), \quad t \in I.$$

This equation reduces to a continuous-time dynamical system if, on I, $g \equiv 1$.

Steady-state solutions: solutions of the form $y(t) \equiv y^*$, $t \in I$ with

$$f\big(y^*\big) = 0.$$

Non-constant solutions: along any *non*-constant solution y,

$$\forall\, t \in I,\ f\big(y(t)\big) \neq 0,$$

hence, for any $t_0 \in I$, $y_0 \in \mathbb{R}$, and $c \in \mathbb{R}$,

$$y(t) = \Phi^{-1}\left(\int_{t_0}^{t} g(\tau)\, d\tau + c\right), \quad t \in I, \quad \text{where} \quad \Phi(y) := \int_{y_0}^{y} \frac{1}{f(\zeta)}\, d\zeta, \quad y \in \mathbb{R},$$

whenever Φ is *well defined* and (*locally*) *invertible*.

Stability theory for scalar continuous-time dynamical systems

Equilibrium points, Lyapunov stability, and asymptotic stability

Assume that I is *unbounded to the right*. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be *locally Lipschitz continuous*, and consider the scalar continuous-time dynamical system

$$y'(t) = f\big(y(t)\big), \quad t \in I.$$

Equilibrium points (or *equilibria*): zeros of f, namely $y^* \in \mathbb{R}$ such that

$$f\left(y^*\right) = 0.$$

Equivalently, $y(t) \equiv y^*$, $t \in I$ is a steady-state solution.

Stable equilibria in the sense of *Lyapunov*: equilibrium points y^* such that sufficiently close solutions remain in a neighborhood of y^* for all forward times $t \in I$ for which they are defined.

Attractive equilibria (or *asymptotically stable* equilibria): stable equilibria y^* such that sufficiently close solutions converge to y^* as $t \longrightarrow \infty$, whenever defined.

Equilibrium points that are *not* Lyapunov stable can be categorized as follows.

- **Unstable equilibria** (or *repulsive* equilibria): equilibria y^* such that all sufficiently close solutions diverge from y^* as $t \longrightarrow \infty$, whenever defined.
- **Semistable equilibria**: equilibria y^* that are attractive on one side (of y^*) and repulsive on the other (whenever the solutions are defined).

Phase-line stability criterion: given an equilibrium point y^*,

- $\begin{cases} f(y) > 0, & \text{as } y \uparrow y^*, \\ f(y) < 0, & \text{as } y \downarrow y^*, \end{cases} \quad \Longrightarrow \quad y^*$ is attractive;
- $\begin{cases} f(y) < 0, & \text{as } y \uparrow y^*, \\ f(y) > 0, & \text{as } y \downarrow y^*, \end{cases} \quad \Longrightarrow \quad y^*$ is unstable;
- if f does *not* change sign in a neighborhood of y^*, then y^* is semistable.

Let $y^* \in \mathbb{R}$ be an equilibrium point and assume f to be *differentiable at* y^*. Then the *first-order Taylor expansion* of f *at* y^* yields the approximation

$$y'(t) \approx f'(y^*)\big(y(t) - y^*\big)$$

for all $t \in I$ such that $y(t)$ remains in a small enough neighborhood of y^*.

Linearized stability test:

- if $f'(y^*) < 0$, then y^* is attractive;
- if $f'(y^*) > 0$, then y^* is unstable;
- if $f'(y^*) = 0$, then y^* is semistable, or its stability requires *higher-order* analysis.

Local linearized solution: as a first-order approximation *near* y^*, for any $c \in \mathbb{R}$,

$$y(t) \approx y^* + c\,e^{f'(y^*)t}.$$

Given a *continuously differentiable* function $U\colon \mathbb{R} \to \mathbb{R}$ (*potential energy*), we consider

$$y'(t) = -U'\big(y(t)\big), \quad t \in I.$$

Equivalently, trajectories $y(t),\ t \in I$ evolve along the *steepest descent* of U.

Energy dissipation: U decreases along trajectories $y(t),\ t \in I$, since

$$\frac{d}{dt} U\big(y(t)\big) \equiv -\big[U'\big(y(t)\big)\big]^2 \leq 0, \quad t \in I.$$

Equilibrium points: the critical (or stationary) points of U, namely $y^* \in \mathbb{R}$ such that

$$U'\big(y^*\big) = 0.$$

Stability classification: given an equilibrium point y^*,

- if y^* is a *strict local minimum* of U, then y^* is attractive;
- if y^* is a *strict local maximum* of U, then y^* is unstable;
- otherwise, y^* is semistable, or its stability requires *higher-order* analysis.

Let $d \in \mathbb{N}$ with $d \geq 2$. Given $\boldsymbol{A} = (a_{ij})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}$ and $\boldsymbol{b} = [b_1, \dots, b_d]^\top \colon I \to \mathbb{R}^d$ (componentwise) *locally integrable*, we consider

$$\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{b}(t), \quad t \in I,$$

where

$$\boldsymbol{y} = [y_1, \dots, y_d]^\top \colon I \to \mathbb{R}^d.$$

Equivalently, for any $i = 1, \dots, d$,

$$y_i'(t) = \boldsymbol{A}_i\,\boldsymbol{y}(t) + b_i(t) \equiv \sum_{j=1}^{d} a_{ij}\,y_j(t) + b_i(t), \quad t \in I,$$

with $\boldsymbol{A}_i := [a_{i1}, \dots, a_{id}]$ the i-th *row* of $\boldsymbol{A}$.

Systems of first-order linear ODEs with constant coefficients

Diagonalizable case: coordinate decoupling via eigenbasis

Assume that $\boldsymbol{A}$ admits d *real eigenvalues* $\lambda_1, \ldots, \lambda_d$, namely the *roots* of its *characteristic polynomial* $\det\big(\boldsymbol{A} - \lambda \boldsymbol{I}_d\big) = 0$ (with $\boldsymbol{I}_d$ the $d \times d$ identity matrix), and that it is *diagonalizable*.

This holds, for instance, when $\boldsymbol{A}$ is *symmetric*, or when its *eigenvalues* are (real and) *distinct*.

Thus, let $\boldsymbol{D} \in \mathbb{R}^{d\times d}$ be the *diagonal matrix* with diagonal entries λ_j, $j = 1, \ldots, d$, and let $\boldsymbol{V} \in \mathbb{R}^{d\times d}$ be the matrix whose *columns* are the corresponding d *eigenvectors* $\boldsymbol{v}^j \equiv \boldsymbol{v}_{\lambda_j}$, $j = 1, \ldots, d$ of $\boldsymbol{A}$ (i.e., $\boldsymbol{A}\boldsymbol{v}^j = \lambda_j \boldsymbol{v}^j$), so that

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{D}\boldsymbol{V}^{-1}.$$

Therefore, if $\boldsymbol{z} := \boldsymbol{V}^{-1}\boldsymbol{y}$, then

$$\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{b}(t), \quad t \in I \quad \Longleftrightarrow \quad \boldsymbol{z}'(t) = \boldsymbol{D}\boldsymbol{z}(t) + \boldsymbol{V}^{-1}\boldsymbol{b}(t), \quad t \in I.$$

That is, for any $i, j = 1, \ldots, d$, $t, t_0 \in I$, and $c_j \in \mathbb{R}$,

$$y_i(t) = \boldsymbol{V}_i\,\boldsymbol{z}(t), \quad z_j(t) = e^{\lambda_j t}\left[\int_{t_0}^{t} \big(\boldsymbol{V}^{-1}\big)_j \boldsymbol{b}(\tau)\, e^{-\lambda_j \tau} d\tau + c_j\right].$$

Systems of first-order linear ODEs with constant coefficients

Continuous-time linear dynamical systems

In the *autonomous case*, given $\boldsymbol{b} \in \mathbb{R}^d$,

$$\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{b}, \quad t \in I$$

$$\Updownarrow$$

$$y_i(t) = \sum_{\substack{j=1,\dots,d \\ \lambda_j = 0}} (\boldsymbol{v}^j)_i \left[(\boldsymbol{V}^{-1})_j \boldsymbol{b}\, t + c_j \right] + \sum_{\substack{j=1,\dots,d \\ \lambda_j \neq 0}} (\boldsymbol{v}^j)_i \left[-\frac{1}{\lambda_j} (\boldsymbol{V}^{-1})_j \boldsymbol{b} + c_j e^{\lambda_j t} \right], \quad t \in I,$$

for any $i = 1, \dots, d$ and $c_1, \dots, c_d \in \mathbb{R}$.

The solution is a *linear combination* of *affine* and *exponential functions*, and its *local* and *asymptotic behaviour* is governed by the *exponential terms* via the *sign* of the *eigenvalues*.

Nonsingular case: when $\det \boldsymbol{A} \neq 0$, that is, $\lambda_j \neq 0$ for all $j = 1, \dots, d$,

$$y_i(t) = (-\boldsymbol{A}^{-1}\boldsymbol{b})_i + \sum_{j=1}^{d} c_j e^{\lambda_j t} (\boldsymbol{v}^j)_i, \quad t \in I,$$

for any $i = 1, \dots, d$ and $c_1, \dots, c_d \in \mathbb{R}$.

Stability theory for continuous-time dynamical systems

Equilibria, stability, instability, and saddle–points

Let $\boldsymbol{f} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be (componentwise) *locally Lipschitz continuous*, and consider

$$\boldsymbol{y}'(t) = \boldsymbol{f}\big(\boldsymbol{y}(t)\big), \quad t \in I.$$

Equilibrium points (*equilibria*): zeros of $\boldsymbol{f}$, namely $\boldsymbol{y}^* \in \mathbb{R}^d$ such that

$$\boldsymbol{f}\big(\boldsymbol{y}^*\big) = \mathbf{0}.$$

Stable, **attractive**, and **unstable/repulsive equilibria**: as in the one-dimensional case.

Saddle equilibria (or *minimax* equilibria): equilibria $\boldsymbol{y}^*$ with both locally attracting and locally repelling trajectories in every neighborhood of $\boldsymbol{y}^*$.

Remark: given an equilibrium point $\boldsymbol{y}^*$, the *sign* analysis of

$$(\boldsymbol{y} - \boldsymbol{y}^*)^\top \boldsymbol{f}(\boldsymbol{y}) \quad \text{as} \quad \boldsymbol{y} \to \boldsymbol{y}^*$$

captures only the local *radial* behaviour of the flow and *cannot* provide a full *phase-space* stability criterion (in *contrast* with the one-dimensional setting).

Stability theory for continuous-time dynamical systems

The linearized stability test

Let $\boldsymbol{y}^* \in \mathbb{R}^d$ be an equilibrium point and assume $\boldsymbol{f} = \left[f_1, \ldots, f_d\right]^\top$ to be *differentiable at* $\boldsymbol{y}^*$. Then the *first-order Taylor expansion* of $\boldsymbol{f}$ *at* $\boldsymbol{y}^*$ yields the approximation

$$\boldsymbol{y}'(t) \approx \boldsymbol{J_f}(\boldsymbol{y}^*)(\boldsymbol{y}(t) - \boldsymbol{y}^*)$$

for all $t \in I$ such that $\boldsymbol{y}(t)$ remains in a sufficiently small neighborhood of $\boldsymbol{y}^*$. Here, for any $i = 1, \ldots, d$, $\left(\boldsymbol{J_f}(\boldsymbol{y}^*)\right)_i = \nabla f_i(\boldsymbol{y}^*)^\top$; explicitly, for any $j = 1, \ldots, d$,

$$\left(\boldsymbol{J_f}(\boldsymbol{y}^*)\right)_{ij} = \frac{\partial f_i}{\partial y_j}(\boldsymbol{y}^*).$$

Linearized stability test:

- if all *eigenvalues* of $\boldsymbol{J_f}(\boldsymbol{y}^*)$ have *strictly negative real part*, then $\boldsymbol{y}^*$ is attractive;
- if all *eigenvalues* of $\boldsymbol{J_f}(\boldsymbol{y}^*)$ have *strictly positive real part*, then $\boldsymbol{y}^*$ is repulsive;
- otherwise, $\boldsymbol{y}^*$ is a saddle–point (when *eigenvalues* have *real parts of mixed sign*), or its stability requires *higher-order* analysis (e.g., *centers*, *semistability*, *bifurcations*).

Given a *continuously differentiable* function $U: \mathbb{R}^d \longrightarrow \mathbb{R}$ (*potential energy*), we consider

$$\boldsymbol{y}'(t) = -\nabla U\big(\boldsymbol{y}(t)\big), \quad t \in I.$$

Energy dissipation: $\dfrac{d}{dt} U\big(\boldsymbol{y}(t)\big) \equiv -\big\|\nabla U\big(\boldsymbol{y}(t)\big)\big\|^2 \leq 0$, $t \in I$.

Equilibrium points: the critical points of U, namely $\boldsymbol{y}^* \in \mathbb{R}^d$ such that $\nabla U\big(\boldsymbol{y}^*\big) = \mathbf{0}$.

Stability classification: given an equilibrium point $\boldsymbol{y}^*$,

- if $\boldsymbol{y}^*$ is a *strict local minimum* of U, then $\boldsymbol{y}^*$ is attractive;
- if $\boldsymbol{y}^*$ is a *strict local maximum* of U, then $\boldsymbol{y}^*$ is repulsive;
- otherwise, $\boldsymbol{y}^*$ is a saddle–point, or its stability requires *higher-order* analysis.

Diagonalizable case: when U is *twice* continuously differentiable, $\boldsymbol{J}_{-\nabla U}(\boldsymbol{y}^*) \equiv -\boldsymbol{H}_U(\boldsymbol{y}^*)$ is *symmetric* (by Schwarz's theorem); hence its *eigenvalues* are *real* and it is *diagonalizable*.

Algebraic digression: the sign of real roots of polynomials

Descartes' rule of signs and a coefficient-based criterion

Let $m \in \mathbb{N} \setminus \{0\}$ and $a_{m-1}, \ldots, a_1, a_0 \in \mathbb{R}$. Consider the monic polynomial

$$p_m(\lambda) := \lambda^m + \sum_{k=0}^{m-1} a_k \lambda^k, \quad \lambda \in \mathbb{R},$$

and let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ be the m *roots* of p_m. Then:

- the *number* of *real positive roots* $\lambda_j > 0$, counted with multiplicity, equals the *number* of *sign changes* (ignoring zeros), *modulo* 2, in the sequence $1, a_{m-1}, \ldots, a_1, a_0$;
- the *number* of *real negative roots* $\lambda_j < 0$, counted with multiplicity, equals the *number* of *sign changes* (ignoring zeros), *modulo* 2, in $1, -a_{m-1}, a_{m-2}, \ldots, (-1)^m a_0$.

In particular, when $\lambda_1, \ldots, \lambda_m$ are *real* (i.e., no complex conjugate pairs occur):

- $\lambda_j < 0$ for *all* $j = 1, \ldots, m$ if, and only if, $a_k > 0$ for *all* $k = 0, \ldots, m-1$;
- $\lambda_j > 0$ for *all* $j = 1, \ldots, m$ if, and only if, $(-1)^{m-k} a_k > 0$ for *all* $k = 0, \ldots, m-1$.

Let $d = 2$ and let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the two eigenvalues of

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Then:

$$\operatorname{Tr} \boldsymbol{A} \equiv a_{11} + a_{22} = \lambda_1 + \lambda_2;$$

$$\det \boldsymbol{A} \equiv a_{11} a_{22} - a_{12} a_{21} = \lambda_1 \lambda_2;$$

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}_2\right) = \lambda^2 - (\operatorname{Tr} \boldsymbol{A})\lambda + \det \boldsymbol{A}, \quad \lambda \in \mathbb{C}.$$

Consequently,

$$\lambda_{1,2} = \frac{1}{2}\left[\operatorname{Tr} \boldsymbol{A} \pm \sqrt{(\operatorname{Tr} \boldsymbol{A})^2 - 4 \det \boldsymbol{A}}\right].$$

Two-dimensional continuous-time linear dynamical systems

Stability classification via determinant and trace

- $\lambda_1 = 0$ or $\lambda_2 = 0 \;\Leftrightarrow\; \det \boldsymbol{A} = 0$, and in this case $\operatorname{Tr} \boldsymbol{A}$ equals the (necessarily real) nonzero eigenvalue, if any.

  Phase portrait: *uniform motion* ($\operatorname{Tr} \boldsymbol{A} = 0$); *stable line of equilibria* ($\operatorname{Tr} \boldsymbol{A} < 0$); *unstable line of equilibria* ($\operatorname{Tr} \boldsymbol{A} > 0$).

- $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2 \;\Leftrightarrow\; \det \boldsymbol{A} < \frac{1}{4}(\operatorname{Tr} \boldsymbol{A})^2$, and, when $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, $\operatorname{sgn} \lambda_1 = \operatorname{sgn} \lambda_2 \;\Leftrightarrow\; \det \boldsymbol{A} > 0$, with $\operatorname{sgn} \lambda_1 = \operatorname{sgn}(\operatorname{Tr} \boldsymbol{A})$.

  Phase portrait: *saddle* ($\det \boldsymbol{A} < 0$); *sink* ($\operatorname{Tr} \boldsymbol{A} < 0$); *source* ($\operatorname{Tr} \boldsymbol{A} > 0$).

- $\lambda_1 = \lambda_2 \in \mathbb{R} \;\Leftrightarrow\; \det \boldsymbol{A} = \frac{1}{4}(\operatorname{Tr} \boldsymbol{A})^2$, and $\operatorname{Tr} \boldsymbol{A} = 2\lambda_1$.

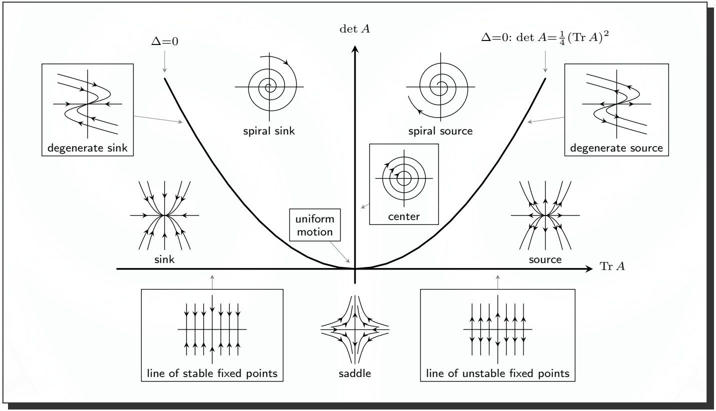
  Phase portrait: *degenerate sink* ($\operatorname{Tr} \boldsymbol{A} < 0$); *degenerate source* ($\operatorname{Tr} \boldsymbol{A} > 0$).

- $\lambda_2 = \bar{\lambda}_1 \in \mathbb{C} \setminus \mathbb{R} \;\Leftrightarrow\; \det \boldsymbol{A} > \frac{1}{4}(\operatorname{Tr} \boldsymbol{A})^2$, and $\operatorname{Tr} \boldsymbol{A} = 2\Re(\lambda_1)$.

  Phase portrait: *center* ($\operatorname{Tr} \boldsymbol{A} = 0$); *spiral sink* ($\operatorname{Tr} \boldsymbol{A} < 0$); *spiral source* ($\operatorname{Tr} \boldsymbol{A} > 0$).

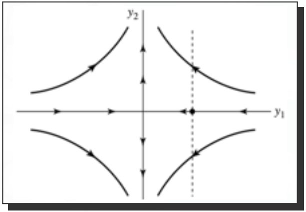
Two-dimensional continuous-time linear dynamical systems

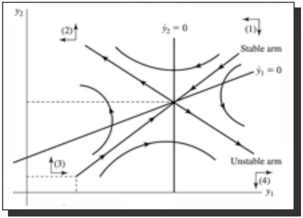
The Poincaré diagram in the trace–determinant plane

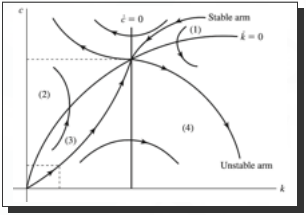


Two-dimensional continuous-time dynamical systems

Example: linear–to–nonlinear transition in local dynamics (saddle case)







Thanks for your attention!