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Subgame-perfect equilibrium strategies in constrained recursive stochastic control problems

Candidate:
Marco Tarsia

Thesis advisor:
Prof.ssa Elisa Mastrogiacomo

Research supervisor:
Prof. Daniele Cassani

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To my loving family

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Introduction

In this Thesis, we study time-inconsistent recursive stochastic control problems where the notion of optimality is defined by means of subgame-perfect equilibrium. In a continuous-time setting, such controls have been introduced in [10] and [14], later completed in [13], and can be thought of as "infinitesimally optimal via spike variation": i.e., they are optimal with respect to a penalty represented by deviations during an infinitesimal amount of time.

In particular, in [14], the authors apply the classical (Pontryagin) maximum principle theory of [45] to deal with the linear Merton portfolio management problem in the context of pseudo-exponential actualization, introducing the concept of subgame-perfect equilibrium policy as a notion to ensure the time-consistency of the portfolio strategy—possibly not unique. They arrive at equivalent formulations in terms of ODEs and integral equations due to the special form of discounting. See also [29], [24], [3], [4], [2], [25] and [43], and, with regard to the deterministic case, [38], [35], [11] and [12].

In this Thesis, we perform similar operations for a more general control problem in the context of recursive utilities and furthermore under a constraint, which we call *state constraint*, and finally we apply our results to the financial sphere.

The theory of recursive optimal control problems in continuous time has attracted much attention in recent years. For the time-consistent framework, we refer in particular to the fundamental works [8] and [16] (see also [17] and references therein). For the time-inconsistent setting, we mention the series of studies by Yong (see, e.g., [40]), whose approach focuses on dynamic programming (Hamilton–Jacobi–Bellman equations).

The approach followed in our work is instead inspired by [14] and [22] and relies on the stochastic maximum principle; see also [31], [32] and [5]. We adapt the classical spike variation technique to obtain a characterization of equilibrium strategies in terms of a generalized Hamiltonian function $\mathcal{H}$ defined through a pair of backward stochastic differential equations.

Our generalized Hamiltonian function, compared with the classical one, contains the driver coefficient of the recursive utility (which has more variables than its analogue in the classical case) and involves a second-order stochastic process.

We emphasize that, in contrast to the classical case, equilibrium strategies are characterized through not only a necessary condition, but also a sufficient condition involving the generalized Hamiltonian even in the absence of extra convexity conditions.

We point out here that the spike variation technique, explicitly required by the definition of equilibrium policy, applies indiscriminately to the case in which the control domain U satisfies particular geometric conditions such as convexity or linearity and to more general cases; see [27] for a different approach, a terminal perturbation method, which is applicable only to the case of time-consistent optimal controls in the classical sense.

An explicit characterization will smoothly be computed for a portfolio problem that we will consider at the end. We would like to observe here also that the assumptions we adopt for the coefficients of our stochastic control system—a decoupled forward-backward stochastic differential equation—are substantially those proposed in [22], although the concept of equilibrium is not used there. See also [1] and [7].

Going further, our analysis is extended to time-inconsistent recursive stochastic control problems under a state constraint defined by means of an additional recursive utility, under appropriate boundedness assumptions. That constraint refers to an expected value, similarly to the one proposed in [45], and so we adapt Ekeland's variational principle (see also [9]) to this more tricky situation.

We use a penalization method, i.e., we consider equilibria for unconstrained problems approximating our constrained problem, and we then apply the previously developed theory to the maximum principle, obtaining corresponding necessary conditions which, passing to the limit, will also give conditions for our original equilibrium.

The existence of approximating equilibria will be guaranteed precisely by Ekeland's variational principle. We also obtain the usual transversality condition, but losing the sign of the first scalar *multiplier* and failing to incorporate the two multipliers into the Hamiltonian from within. A key point here is to know the distance function with respect to a closed subset of $\mathbb{R}^N$.

We would like to point out that this procedure seems to be able to manage a wide variety of constraints (quite different from the one just mentioned). For instance, one could take a pointwise constraint on the state process at the terminal instant as those considered in the time-consistent case in, e.g., [27] and [46] where, however, convexity is assumed and the spike variation technique is not used. Alternatively, one could consider as a constraint an expected value of a function of the terminal state as in [18], where local minimizers, Clarke's tangent cone and adjacent cones of first and second order are used. For further details, we suggest that the reader consult [39], [42], and [41].

Finally, the theoretical results are applied in the financial field to finite horizon investment-consumption policies with non-exponential actualization (e.g., a hyperbolic one). In particular, we extend the results contained in the aforementioned works [10], [13], and [14] in two directions: first by introducing the recursive utilities and second by considering state constraints.

We emphasize that these problems are still far from being fully developed. We refer to [34] as one of the first notable works in portfolio choice theory with constant-relative-risk aversion (CRRA)-type preferences, for a convex and compact constraint defined through a (pseudo) risk measure such as value at risk (VaR) on a wealth process at a future time instant "very close" to the present. Here, the market coefficients are random but independent of the Brownian motion driving the stocks. For a generalization, see [30] (CRRA preferences) and also [23] and [6] (martingale methods). For further directions in this field, see [15] (bond portfolios) and [21, 20] (optimal derivative design), among the others.

We stress that, under appropriate hypotheses, our results cover the case where the constraint is, more specifically, a risk constraint, i.e., the additional recursive utility derives from a suitable dynamic risk measure defined by means of a g-expectation as, e.g., in [37] and references therein (this is an approach that deserves further investigation).

We seek here controls in feedback form, partially mimicking what is done in [14] (or [13]), where explicit calculations are feasible. The shape of the recursive utilities, in both the unconstrained and constrained cases, follows the classic Uzawa type (see [8]), but many other choices are possible. See also [19], [26], [44] and [33].

The Thesis is organized as follows. In Chapter 1, we formulate the notion of (subgame-perfect) equilibrium policy and the two classes of problems we are interested in, namely, unconstrained and constrained (a matter of obtaining a maximum principle). In Section 1.1, we introduce the basic notations and spaces of stochastic processes we work with. In Section 1.2, we introduce: the control domain

$$U;$$

the space

$$\mathscr{U}[t,T]$$

of admissible controls $\boldsymbol{u}(\cdot)$ (see Definition 1.2.1); the state domain

$$I;$$

the admissible 4-tuple

$$(\boldsymbol{u}(\cdot), X(\cdot), Y(\cdot\,; t), Z(\cdot\,; t)),$$

solution of the recursive stochastic control problem

$$\begin{cases} dX(s) = b(s, X(s), \boldsymbol{u}(s))ds + \sigma(s, X(s), \boldsymbol{u}(s))dW(s), \\ dY(s;t) = -f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t);t)ds + Z(s;t)dW(s), \\ X(t) = x, \quad Y(T;t) = h(X(T);t), \end{cases}$$

(with $x \in I$, $T \in]0, \infty[$ and $s \in [t, T]$), where $X(\cdot)$ is the admissible state process (see Definition 1.2.3) and $Y(\cdot\,;t)$ is the admissible recursive utility (see Definition 1.2.4); the utility, or cost, functional

$$J(\boldsymbol{u}(\cdot); t, x) \doteq Y(t;t)$$

(see Definition 1.2.6).

Therefore, we present the notions of equilibrium policy

$$\boldsymbol{\Pi},$$

equilibrium pair and equilibrium 4-tuple by relating them to the behavior, roughly speaking, of lim inf of the form

$$\liminf_{\varepsilon \downarrow 0} \frac{J(\bar{\boldsymbol{u}}^{\varepsilon}(\cdot); t, x) - J(\bar{\boldsymbol{u}}(\cdot); t, x)}{\varepsilon}$$

(see Definition 1.2.7).

The first stochastic control problem, the unconstrained one, consists in finding necessary and sufficient conditions for an equilibrium 4-tuple (Problem 1).

We also add the (one-dimensional) state constraint

$$\boldsymbol{J}(\boldsymbol{u}(\cdot); t, x) = \boldsymbol{Y}(t;t) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$$

($\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}} \subset \mathbb{R}$), where $(\boldsymbol{u}(\cdot), X(\cdot))$ is an equilibrium pair and

$$\begin{cases} d\boldsymbol{Y}(s;t) = -\boldsymbol{f}(s, X(s), \boldsymbol{u}(s), \boldsymbol{Y}(s;t), \boldsymbol{Z}(s;t);t)ds + \boldsymbol{Z}(s;t)dW(s), \\ \boldsymbol{Y}(T;t) = \boldsymbol{h}(X(T);t), \end{cases}$$

(see Definition 1.2.9).

The second stochastic control problem, the constrained one, consists in finding necessary conditions for an equilibrium pair that satisfies the state constraint (Problem 2). In Section 1.3, we present some preliminary results mainly on continuous dependence in BSDEs theory and in metric space theory (see, above all, Lemma 1.3.1 and Lemma 1.3.3).

In Chapter 2, we present necessary and sufficient conditions for the existence of an equilibrium policy in the unconstrained case. In Section 2.1, we define the adjoint equations/processes of first order and second order (associated with an admissible 4-tuple), respectively

$$\begin{cases} dp(s;t) = -g(s,p(s;t),q(s;t);t)ds + q(s;t)dW(s), \\ p(T;t) = h_x(\bar{X}(T);t), \end{cases}$$

and

$$\begin{cases} dP(s;t) = -G(s,P(s;t),Q(s;t);t)ds + Q(s;t)dW(s), \\ P(T;t) = h_{xx}(\bar{X}(T);t), \end{cases}$$

where the maps

$$g\colon [0,T]_s \times \mathbb{R}_p \times \mathbb{R}_q \times [0,T[_t \to \mathbb{R},$$

and

$$G\colon [0,T]_s \times \mathbb{R}_P \times \mathbb{R}_Q \times [0,T[_t \to \mathbb{R},$$

respectively linear in (p,q) and (P,Q) and explained later, descend from a second-order expansion of the cost functional and so reflect the structure of the functional itself given in terms of recursive utility (see Definition 2.1.2 and Definition 2.1.3). The generalized Hamiltonian (function) of first order (associated with b,σ,f) is

$$\begin{aligned} H(s,x,\boldsymbol{u},y,z,p,q;t,\bar{x},\bar{\boldsymbol{u}}) &= pb(s,x,\boldsymbol{u}) + q\sigma(s,x,\boldsymbol{u}) \\ &\quad + f(s,x,\boldsymbol{u},y,z+p[\sigma(s,x,\boldsymbol{u})-\sigma(s,\bar{x},\bar{\boldsymbol{u}})];t) \end{aligned}$$

(see Definition 2.1.4) and the generalized Hamiltonian (function) of second order (associated with b,σ,f) is

$$\begin{aligned} \mathcal{H}(s,x,\boldsymbol{u},y,z,p,q,P;t,\bar{x},\bar{\boldsymbol{u}}) &= H(s,x,\boldsymbol{u},y,z,p,q;t,\bar{x},\bar{\boldsymbol{u}}) \\ &\quad + \tfrac{1}{2}P[\sigma(s,x,\boldsymbol{u})-\sigma(s,\bar{x},\bar{\boldsymbol{u}})]^2 \end{aligned}$$

(see Definition 2.1.5). Finally, we present a preparatory lemma, Lemma 2.1.1, and finally the maximum principle, Theorem 2.1.1, that solves Problem 1. In Section 2.2, we prove Lemma 2.1.1 leaning on the classical "Taylor expansions" as in [45, Chap. 3, Sect. 4], and then a more recent technical procedure as in [22], to get a further expansion of first order of $J(\,\boldsymbol{\cdot}\,;t,x)$ through a family of approximating linear BSDEs and a useful change of numéraire

$$\kappa(\,\boldsymbol{\cdot}\,;t)$$

(see Definition 2.1.1). In Section 2.3, we extend the results developed in the previous sections to the multidimensional case. Here the most interesting aspect lies in the modification of the cost functional (and, consequently, of the maximum principle condition) as

$$J(\boldsymbol{u}(\cdot);t,x) \doteq \gamma(Y(t;t);t)$$

where

$$\gamma\colon \mathbb{R}^m_y \times [0,T[_t \to \mathbb{R}$$

is of (differentiability) class C^1 w.r.t. the variable $y \in \mathbb{R}^m$. In Section 2.4, we propose a significant application to portfolio management which involves a linear recursive stochastic control problem, where any admissible control is an investment-consumption strategy, the state process is a wealth process and the recursive utility is of Uzawa type:

$$\begin{cases} dX(s) = X(s)\big[\big(r + \mu\zeta(s) - c(s)\big)\,ds + \sigma\zeta(s)\,dW(s)\big], \\ dY(s;t) = -\hbar(s;t)\big[-\upsilon(c(s)X(s)) - \beta(s;t)Y(s;t) - \gamma(s;t)Z(s;t)\big]\,ds \\ \qquad\qquad + Z(s;t)\,dW(s), \\ X(t) = x, \quad Y(T;t) = -\hat{\hbar}(T;t)\hat{\upsilon}(X(T)), \end{cases}$$

(with $x > 0$), where

$$\upsilon(\cdot), \hat{\upsilon}(\cdot)$$

are two scalar functions of a real variable that satisfy the classical Uzawa–Inada conditions and

$$\hbar(\cdot\,;t), \hat{\hbar}(\cdot\,;t)$$

are two discount functions on $[t,T]$ (possibly non-exponential).

In Chapter 3, we present necessary conditions for the constrained problem, i.e. for an equilibrium pair $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot))$ that satisfies the state constraint. Assuming that U is bounded, Problem 2 is solved in Theorem 3.1.1 by applying a penalty method based on Ekeland's variational principle (Lemma 3.2.1) in an analogous way to the classical case in [45, Chap. 3, Sect. 6], but through a more complex proof, since there could be an admissible control $\boldsymbol{u}(\cdot)$ with

$$J(\boldsymbol{u}(\cdot);t,x) < J(\bar{\boldsymbol{u}}(\cdot);t,x),$$

hence the denominators of some ratios may become zero.

In Section 3.3, we solve Problem 2 under the assumption that $\bar{\boldsymbol{u}}(\cdot)$ is essentially bounded (as a stochastic process), but with U not necessarily bounded, applying a very natural approximation method (see Corollary 3.3.1). In Section 3.4, similarly to Section 2.4, we apply the results obtained in the previous sections to portfolio management by adding a recursive utility system where the recursive utility is of Uzawa type as well, to get a state constraint:

$$\begin{cases} d\boldsymbol{Y}(s;t) = \boldsymbol{\hbar}(s;t)\boldsymbol{\beta}(s;t)\boldsymbol{Y}(s;t)ds + \boldsymbol{Z}(s;t)dW(s), \\ \boldsymbol{Y}(T;t) = -\hat{\boldsymbol{\hbar}}(T;t)\hat{\boldsymbol{v}}(X(T)). \end{cases}$$

Possible future research directions are discussed in the brief concluding chapter, concerning the investigation of more general state constraints or with more general cost functionals or different concrete applications.

Chapter 1

Problem formulation

1.1 Notations

Set $T \in \,]0, \infty[$ as a finite deterministic *horizon* and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space such that we can define a one-dimensional Brownian motion, or Wiener process, $W = (W(t))_{t\in[0,T]}$ on it. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the completed filtration generated by W, for which we suppose

$$\mathcal{F}_T = \mathcal{F}$$

(system noise is the only source of uncertainty in the problem). Thus, the filtered space

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$$

satisfies the usual conditions. In this regard, see, e.g., [45, Chap. 1, Sect. 2].

Remark 1.1.1. For any non-empty set $\mathcal{I}$ of indices $\imath$, we will keep implicit the dependence on the sample variable $\omega \in \Omega$ for each stochastic process on $\mathcal{I} \times \Omega$, as is usually done (and as indeed we have just done for W). We specify also that any stochastic process on $\mathcal{I} \times \Omega$ must be seen as its equivalence class given by the quotient with respect to the equivalence relation $\sim$ of indistinguishability: i.e., for any process $X = (X(\imath))_{\imath\in\mathcal{I}}$, $\tilde{X} = (\tilde{X}(\imath))_{\imath\in\mathcal{I}}$ on $\mathcal{I} \times \Omega$, $X \sim \tilde{X}$ if and only if

$$\mathbf{P}\big[\forall\, \imath \in \mathcal{I},\, X(\imath) = \tilde{X}(\imath)\big] = 1.$$

We introduce the following, rather familiar, notation, in which $E \in \mathcal{B}(\mathbb{R})$, $d \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $V \subseteq \mathbb{R}^d = \mathbb{R}^{d\times 1}$ is a vector subspace, $t, \tau \in [0, T]$ with $t \neq T$, and $p \in [1, \infty[$:

- $\lesssim$. Less than or equal to, unless there are positive multiplicative constants (independent of what is involved) about which we are not particularly interested in being more explicit.

- $\mathbf{m}[\,\cdot\,]$. The one-dimensional Lebesgue measure on $\mathcal{B}(\mathbb{R})$.
- $\mathbb{1}_E(\cdot)$. The indicator function of the set E, i.e., for any $x \in \mathbb{R}$,

$$\mathbb{1}_E(x) \coloneqq \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

- $\mathbf{E}[\,\cdot\,]$. The expected value w.r.t. $\mathbf{P}$ of a ($\mathbf{P}$-integrable) real-valued $\mathcal{F}$-measurable random variable X on Ω, i.e.,

$$\mathbf{E}[X] \equiv \mathbf{E}\,X \coloneqq \int_\Omega X(\omega)\,d\mathbf{P}(\omega) \in \mathbb{R}.$$

- $L^p_\tau(\Omega; V)$. The (Banach) space of V-valued $\mathcal{F}_\tau$-measurable random variables X on Ω such that

$$\|X\|_p^p \coloneqq \mathbf{E}\,|X|^p < \infty.$$

- $L^\infty_\tau(\Omega; V)$. The space of ($\mathbf{P}$-a.s.) bounded V-valued $\mathcal{F}_\tau$-measurable random variables X on Ω, i.e., with

$$\|X\|_\infty \coloneqq \inf\{\, K \in [0,\infty[\mid |X| \le K \ \mathbf{P}\text{-a.s.}\,\} < \infty$$

  where eventually, by convention,

$$\inf \emptyset = \infty.$$

- $\mathcal{L}^p_\mathbb{F}(t,T;V)$. The space of V-valued $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable processes $X = (X(s))_{s\in[t,T]}$ on $[t,T]\times\Omega$ (or $X(\cdot)$, for short) such that

$$\|X(\cdot)\|_p^p \equiv \|X\|_p^p \coloneqq \mathbf{E}\int_t^T |X(s)|^p\,ds < \infty.$$

- $\mathcal{L}^\infty_\mathbb{F}(t,T;V)$. The space of bounded V-valued $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable processes $X = (X(s))_{s\in[t,T]}$ on $[t,T]\times\Omega$, i.e., with

$$\|X(\cdot)\|_\infty \equiv \|X\|_\infty \coloneqq \inf\Big\{ K \in [0,\infty[\ \Big|\ \sup_{s\in[t,T]}|X(s)| \le K \ \mathbf{P}\text{-a.s.}\Big\} < \infty.$$

- $\mathcal{L}^p_\mathbb{F}(\Omega; \mathcal{C}([t,T];V))$. The space of V-valued $(\mathcal{F}_s)_{s\in[t,T]}$-adapted ($\mathbf{P}$-a.s.) continuous processes $X = (X(s))_{s\in[t,T]}$ on $[t,T]\times\Omega$ such that

$$\|X(\cdot)\|_{\mathcal{C},p}^p \equiv \|X\|_{\mathcal{C},p}^p \coloneqq \mathbf{E}\sup_{s\in[t,T]}|X(s)|^p < \infty.$$

Remark 1.1.2. For $X = (X(s))_{s\in[t,T]} \in \mathcal{L}^p_{\mathbb{F}}(t,T;V)$,

$$\mathbf{E}\left(\int_t^T |X(s)|ds\right)^p \leq \|X(\cdot)\|_p^p < \infty$$

(by the classical Jensen's inequality, w.r.t. the Lebesgue measure on $[t,T]$, for a.a. fixed $\omega \in \Omega$).

Remark 1.1.3. A $\mathbb{R}^d$-valued $(\mathcal{F}_s)_{s\in[t,T]}$-adapted process $X = (X(s))_{s\in[t,T]}$ on $[t,T]\times\Omega$ admits an $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable modification (stochastically equivalent process) and, if X is ($\mathbf{P}$-a.s.) left continuous or right continuous as a process, then X itself is $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable. See, e.g., [45, Chap. 1, Sect. 2].

1.2 Definitions and assumptions

Take $n \in \mathbb{N}^*$ and $\mathbb{R}^n$ equipped with the Euclidean topology and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ with its Lebesgue measure, as will be the case for any other Euclidean space, and choose a *control domain*

$$U \in \mathcal{B}(\mathbb{R}^n) \setminus \{\emptyset\}$$

(not necessarily bounded, for now).

Definition 1.2.1 (Admissible control)**.** Fix an arbitrary *initial instant*

$$t \in [0,T[.$$

For an appropriate

$$p \in [2,\infty[$$

that we do not want to give in explicit form (see [22, Introduction]), we set

$$\mathscr{U}[t,T] \doteq \{\, \boldsymbol{u}(\cdot) \in \mathcal{L}^p_{\mathbb{F}}(t,T;\mathbb{R}^n) \mid \boldsymbol{u}(\cdot) \textit{ is } U\textit{-valued} \,\} \tag{1.1}$$

and we call an *admissible control* any element $\boldsymbol{u}(\cdot)$ of $\mathscr{U}[t,T]$.

Remark 1.2.1. If U is bounded (in $\mathbb{R}^n$), then the class $\mathscr{U}[t,T]$ in (1.1) simply coincides with the one constituted by the U-valued processes $\boldsymbol{u}(\cdot)$ on $[t,T]\times\Omega$ such that $\boldsymbol{u}(\cdot)$ is $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable.

Definition 1.2.2 (Spike variation)**.** Fix $t \in [0,T[$, $\bar{\boldsymbol{u}}(\cdot) \in \mathscr{U}[t,T]$, $\varepsilon \in]0,T-t[$, and a set

$$E_t^\varepsilon \in \mathcal{B}([t,T])$$

with length
$$|E_t^\varepsilon| \coloneqq \mathbf{m}[E_t^\varepsilon] = \varepsilon.$$

For $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$, we call the *spike* (or *needle*) *variation of* $\bar{\boldsymbol{u}}(\cdot)$ *w.r.t.* $\boldsymbol{u}(\cdot)$ *and* E_t^ε the admissible control $\bar{\boldsymbol{u}}^\varepsilon(\cdot) \in \mathscr{U}[t,T]$ defined by setting

$$\bar{\boldsymbol{u}}^\varepsilon \doteq \bar{\boldsymbol{u}} + (\boldsymbol{u} - \bar{\boldsymbol{u}})\mathbb{1}_{E_t^\varepsilon}. \tag{1.2}$$

Remark 1.2.2. The spike variation $\bar{\boldsymbol{u}}^\varepsilon(\cdot)$ in (1.2) is explicitly given, for any $s \in [t,T]$, by

$$\bar{\boldsymbol{u}}^\varepsilon(s) = \begin{cases} \bar{\boldsymbol{u}}(s) & \text{if } s \in [t,T] \setminus E_t^\varepsilon, \\ \boldsymbol{u}(s) & \text{if } s \in E_t^\varepsilon. \end{cases}$$

Notation 1.2.1. Any alphabetic letter appearing as a subscript of a prescribed set that appears explicitly as (part of) the domain of a function such as, among others, $\boldsymbol{u} = (u_1, \ldots, u_n)^\mathsf{T}$ for
$$U = U_{\boldsymbol{u}}$$
or x for
$$\mathbb{R} = \mathbb{R}_x$$
should be seen as our preferred notation for the generic variable element of that domain.

Fix deterministic maps

$$b, \sigma \colon [0,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}} \to \mathbb{R},$$

$$f \colon [0,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}} \times \mathbb{R}_y \times \mathbb{R}_z \times [0,T[_t \to \mathbb{R},$$

and
$$h \colon \mathbb{R}_x \times [0,T[_t \to \mathbb{R}$$
such that the following assumption holds, similarly to [22] (to which we refer to better understand why such strong assumptions are needed).

Assumption 1. The maps b, σ, f, h are continuous w.r.t. all their variables and, for any $t \in [0,T[$, there exists $L_t \in]0,\infty[$ such that, whatever map

$$\varphi(s,x,\boldsymbol{u},y,z;t)$$

between $b(s,x,\boldsymbol{u})$, $\sigma(s,x,\boldsymbol{u})$, $f(s,x,\boldsymbol{u},y,z;t)$, and $h(x;t)$ is taken, and for any $s \in [t,T]$, $\boldsymbol{u} \in U$ and $x,y,z \in \mathbb{R}$,

$$|\varphi(s,x,\boldsymbol{u},y,z;t)| \leq L_t\big(1 + |x| + |\boldsymbol{u}| + |y| + |z|\big).$$

Next, b, σ, h are of (differentiability) class C^2 w.r.t. the variable $x \in \mathbb{R}$; $b_x, b_{xx}, \sigma_x, \sigma_{xx}$ are bounded (on $[t,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}}$) and continuous w.r.t. $(x, \boldsymbol{u}) \in \mathbb{R} \times U$; h_x, h_{xx} are bounded and continuous (on $\mathbb{R}_x$); $f(\cdot\,; t)$ is of class C^2 w.r.t. $(x, y, z) \in \mathbb{R}^3$, with $\mathrm{D}f(\cdot\,;t)$ and $\mathrm{D}^2 f(\cdot\,;t)$ (gradient and Hessian matrix of $f(\cdot\,;t)$ w.r.t. (x,y,z) respectively) being bounded (on $[t,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}} \times \mathbb{R}_y \times \mathbb{R}_z$) and continuous w.r.t. $(x, \boldsymbol{u}, y, z) \in \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R}$.

Remark 1.2.3. Regarding Assumption 1, we point out the following.

- The relations with $\varphi = b$ and $\varphi = \sigma$ could really depend on t through the fact that $s \in [t, T]$.

- The conditions of sublinear growth and boundedness imply something that is somehow stronger than implied by the classic conditions in [45, Chap. 3, Sect. 3]: more precisely, for any $t \in [0, T[$, there exists $L_t \in \,]0, \infty[$ such that, whatever map $\varphi(s, x, \boldsymbol{u}, y, z; t)$ between $b(s, x, \boldsymbol{u})$, $\sigma(s, x, \boldsymbol{u})$, $f(s, x, \boldsymbol{u}, y, z; t)$, and $h(x; t)$ is taken, for any $s \in [t, T]$, $\boldsymbol{u}, \hat{\boldsymbol{u}} \in U$, and $x, \hat{x}, y, \hat{y}, z, \hat{z} \in \mathbb{R}$,

$$\begin{aligned}
&|\varphi(s, x, \boldsymbol{u}, y, z; t) - \varphi(s, \hat{x}, \hat{\boldsymbol{u}}, \hat{y}, \hat{z}; t)| \\
&\qquad \vee |\varphi_x(s, x, \boldsymbol{u}, y, z; t) - \varphi_x(s, \hat{x}, \hat{\boldsymbol{u}}, \hat{y}, \hat{z}; t)| \\
&\qquad\qquad \vee |\varphi_{xx}(s, x, \boldsymbol{u}, y, z; t) - \varphi_{xx}(s, \hat{x}, \hat{\boldsymbol{u}}, \hat{y}, \hat{z}; t)| \\
&\qquad\qquad\qquad \le L_t\big(|x - \hat{x}| + |\boldsymbol{u} - \hat{\boldsymbol{u}}| + |y - \hat{y}| + |z - \hat{z}|\big)
\end{aligned}$$

  and

$$|\varphi(s, 0, \boldsymbol{u}, 0, 0; t)| \le L_t.$$

- A similar condition could work even if, relative at least to the variable $\boldsymbol{u} \in U$ or $x \in I$, it uses a more generic modulus of continuity $\bar{\omega}(\cdot)$ than a linear one, namely, a map

$$\bar{\omega} \colon [0, \infty[\,\to [0, \infty[$$

  that is non-decreasing, has $\lim_{\delta \downarrow 0} \bar{\omega}(\delta) = \bar{\omega}(0) = 0$ and is such that it quantitatively measures the uniform continuity of some (continuous) function between metric spaces

$$\eta \colon (V, \mathrm{d}_V) \to (\bar{V}, \mathrm{d}_{\bar{V}})$$

  in the sense that, for any $\boldsymbol{v}, \hat{\boldsymbol{v}} \in V$,

$$\mathrm{d}_{\bar{V}}(\eta(\boldsymbol{v}), \eta(\hat{\boldsymbol{v}})) \le \bar{\omega}(\mathrm{d}_V(\boldsymbol{v}, \hat{\boldsymbol{v}})).$$

- See Section 2.4 in Chapter 2 for a situation where the maps b, σ, f and h satisfy all the regularity conditions required by Assumption 1.

Now, choose a *state domain*

$$I \subseteq \mathbb{R}$$

that is a non-empty open interval. For fixed $t \in [0, T[$, all the following stochastic differential equations and corresponding (adapted) solutions are taken on

$$[t, T] \times \Omega.$$

Definition 1.2.3 (Admissible state process)**.** Fix $t \in [0, T[$ and $\boldsymbol{u}(\cdot) \in \mathscr{U}[t, T]$. For an arbitrary fixed *initial state*

$$x \in I$$

we call the *state equation* or *controlled system* (in the *strong formulation*) the forward stochastic differential equation

$$\begin{cases} dX(s) = b(s, X(s), \boldsymbol{u}(s))\,ds + \sigma(s, X(s), \boldsymbol{u}(s))\,dW(s), \\ X(t) = x, \end{cases} \tag{1.3}$$

(where $s \in [t, T]$) and we call an *admissible state process* any solution $X(\cdot)$ of (1.3) that belongs to

$$\mathcal{L}^2_{\mathbb{F}}(t, T; \mathbb{R}).$$

Remark 1.2.4. Regarding Definition 1.2.3, we point out the following.

- The equation (1.3) is a controlled forward stochastic differential equation (FSDE) in Itô differential form, with finite deterministic horizon T and with random coefficients that depend on the sample $\omega \in \Omega$ only through $\boldsymbol{u}(\cdot)$ and $X(\cdot)$ itself, and it depends also on b, σ, t, and x (as well as W).

- The term "strong formulation", which henceforth we will not repeat, alludes to the fact that the filtered space of probability $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is fixed a priori together with W and therefore must not be sought as part of the solution of (1.3) (see, e.g., [45, Chap. 1, Sect. 6]).

- Suppose Assumption 1 holds, at least as regards b and σ. Fix $t \in [0, T[$, $x \in I$, and $\boldsymbol{u}(\cdot) \in \mathscr{U}[t, T]$. Then there exists a unique solution

  $$X(\cdot) \in \mathcal{L}^2_{\mathbb{F}}(\Omega; \mathcal{C}([t, T]; \mathbb{R}))$$

  of the FSDE (1.3) and its Itô integral form is given, for any $s \in [t, T]$, by

  $$X(s) = x + \int_t^s b(r, X(r), \boldsymbol{u}(r))\,dr + \int_t^s \sigma(r, X(r), \boldsymbol{u}(r))\,dW(r). \tag{1.4}$$

  See, e.g., Proposition 1.2.1 below.

Notation 1.2.2. We specify the dependence on the elements involved by writing

$$X(\cdot) = X^{t,x,\boldsymbol{u}}(\cdot) \coloneqq X(\cdot\,; t, x, \boldsymbol{u}(\text{-})).$$

Assumption 2. For $t \in [0, T[$, $\boldsymbol{u}(\cdot) \in \mathscr{U}[t, T]$, $x \in I$, $s \in [t, T]$, and $\mathbf{P}$-a.s.,

$$X^{t,x,\boldsymbol{u}}(s) \in I.$$

Remark 1.2.5. Under Assumption 2, the interval I may depend on T: e.g., the larger T is, the larger I may also be (however, in the worst case scenario, we could always take $I = \mathbb{R}$). Moreover, if we prefer, we can imagine that the domain component in the variable x of the maps b, σ, f and h is restricted precisely to I in such a way that the (analogue of) Assumption 1 still holds. See Section 2.4 in Chapter 2 for a situation where Assumption 2 is satisfied.

Definition 1.2.4 (Admissible recursive utility)**.** For $t \in [0, T[$, $x \in I$ and $\boldsymbol{u}(\cdot) \in \mathscr{U}[t, T]$, let $X(\cdot)$ be an admissible state process as in Definition 1.2.3. We call a *recursive (dis)utility system* a backward stochastic differential equation

$$\begin{cases} dY(s;t) = -f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t); t)\,ds + Z(s;t)\,dW(s), \\ Y(T;t) = h(X(T);t), \end{cases} \tag{1.5}$$

(where $s \in [t, T]$) and we call an *admissible recursive utility* any process $Y(\cdot\,;t)$ such that $(Y(\cdot\,;t), Z(\cdot\,;t))$ is a pair solution of (1.5) that belongs to

$$\mathcal{L}^2_{\mathbb{F}}(\Omega; \mathcal{C}([t,T];\mathbb{R})) \times \mathcal{L}^2_{\mathbb{F}}(t, T; \mathbb{R}).$$

Remark 1.2.6. Regarding Definition 1.2.4, we point out the following.

- The equation (1.5) is a backward stochastic differential equation (BSDE) in Itô differential form, decoupled from the FSDE (1.3) of Definition 1.2.3 on which it totally depends. Here, we have something much more general than a stochastic differential (dis)utility (SDU) in its original meaning: that is, essentially, a BSDE such as

$$\begin{cases} d\Xi(s;t) = -F(s, \boldsymbol{u}(s), \Xi(s;t); t)\,ds + \mathrm{O}(s;t)\,dW(s), \\ \Xi(T;t) = \xi_t, \end{cases}$$

  (where $\xi_t \in L^2_T(\Omega;\mathbb{R})$ and $s \in [t, T]$). See, e.g., [8] and references therein.

- The term "disutility" anticipates the fact that there will be something to be minimized (not maximized). This will be done in a more general sense than the classic one: precisely, in the sense of subgame-perfect equilibrium strategies.

- Suppose Assumption 1 holds and fix $t \in [0,T[$, $x \in I$, and $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$. Then there exists a unique pair solution

$$(Y(\cdot\,;t), Z(\cdot\,;t)) \in \mathcal{L}^2_{\mathbb{F}}(\Omega; \mathcal{C}([t,T];\mathbb{R})) \times \mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R})$$

of the BSDE (1.5), whose Itô integral form is given, for any $s \in [t,T]$, by

$$\begin{aligned} Y(s;t) = h(X(T);t) \\ + \int_s^T f(r, X(r), \boldsymbol{u}(r), Y(r;t), Z(r;t); t)\, dr \\ - \int_s^T Z(r;t)\, dW(r). \end{aligned} \quad (1.6)$$

See Proposition 1.2.1 below and, for everything related to the fundamental theory of BSDEs, see, e.g., [45, Chap. 7] and [17].

Notation 1.2.3. We specify the dependence on the elements involved by writing

$$Y(\cdot\,;t) = Y^{x,\boldsymbol{u}}(\cdot\,;t) \coloneqq Y(\cdot\,;t,x,\boldsymbol{u}(\text{-}))$$

and

$$Z(\cdot\,;t) = Z^{x,\boldsymbol{u}}(\cdot\,;t) \coloneqq Z(\cdot\,;t,x,\boldsymbol{u}(\text{-})).$$

Remark 1.2.7. $Y(t;t)$ is a deterministic constant. Indeed, since x is a deterministic constant, $X(T)$ (see (1.4)) is measurable w.r.t. the completed σ-algebra $\tilde{\mathcal{F}}_{t,T}$ on Ω generated by the process

$$(W(s) - W(t))_{s\in[t,T]},$$

and therefore $Y(t;t)$ (see (1.6)) is simultaneously measurable w.r.t. the two mutually independent σ-algebras $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_{t,T}$ (an argument of this kind is found in, e.g., [7]). Consequently,

$$\begin{aligned} \mathbf{E}\, Y(t;t) &\equiv \mathbf{E}\left[\int_t^T f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t); t)\, ds + h(X(T);t)\right] \\ &= Y(t;t). \end{aligned}$$

On the other hand, it is not possible to establish in general that $Z(t;t)$ is a deterministic constant.

Remark 1.2.8. We will prefer the notation $X(\cdot)$ to other possible notations such as $X(\cdot\,;t)$, but we will retain the notations $Y(\cdot\,;t)$ and $Z(\cdot\,;t)$.

Definition 1.2.5 (Recursive stochastic control problem)**.** Fix $t \in [0,T[$, $x \in I$ and $\boldsymbol{u}(\,\cdot\,) \in \mathscr{U}[t,T]$. We call the *recursive stochastic control problem* the combination of the two stochastic differential equations (1.3) and (1.5), i.e.,

$$\begin{cases} dX(s) = b(s, X(s), \boldsymbol{u}(s))ds + \sigma(s, X(s), \boldsymbol{u}(s))dW(s), \\ dY(s;t) = -f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t);t)ds + Z(s;t)dW(s), \\ X(t) = x, \quad Y(T;t) = h(X(T);t), \end{cases} \tag{1.7}$$

(where $s \in [t,T]$) and, if $(X(\cdot), Y(\cdot\,;t), Z(\cdot\,;t))$ is a solution of (1.7) that belongs to

$$\mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R}) \times \mathcal{L}^2_{\mathbb{F}}(\Omega;\mathcal{C}([t,T];\mathbb{R})) \times \mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R}),$$

then we call $(\boldsymbol{u}(\cdot), X(\cdot), Y(\cdot\,;t), Z(\cdot\,;t))$ an *admissible 4-tuple.*

Remark 1.2.9. The equation/system (1.7) is a controlled decoupled forward-backward stochastic differential equation/system (FBSDE) in Itô differential form and, of course, we could use Notations 1.2.2 and 1.2.3 for the respective components of the corresponding solution.

Regarding the recursive stochastic control problem (1.7) of Definition 1.2.5, the following standard result for existence, uniqueness, and regularity holds (see, e.g., [28]).

Proposition 1.2.1. *Suppose Assumption 1 holds and fix* $t \in [0,T[$*,* $x \in I$ *and* $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$*. Then there exists a unique solution*

$$(X(\cdot), Y(\cdot\,;t), Z(\cdot\,;t)) \in \mathcal{L}^2_{\mathbb{F}}(\Omega;\mathcal{C}([t,T];\mathbb{R}))^2 \times \mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R})$$

of the FBSDE (1.7) *and*

$$\mathbf{E} \sup_{s\in[t,T]} |X(s)|^2 + \mathbf{E} \sup_{s\in[t,T]} |Y(s;t)|^2 + \mathbf{E} \int_t^T |Z(s;t)|^2 ds \lesssim 1 + x^2 + \mathbf{E} \int_t^T |\boldsymbol{u}(s)|^2 ds.$$

Definition 1.2.6 (Utility functional)**.** Suppose Assumption 1 holds, and fix $t \in [0,T[$ and $x \in I$. For any $\boldsymbol{u}(\,\cdot\,) \in \mathscr{U}[t,T]$, consider the solution $(X(\,\cdot\,), Y(\,\cdot\,;t), Z(\,\cdot\,;t))$ of the FBSDE (1.7) as in Proposition 1.2.1. We call *(dis)utility* or *cost functional* the functional $J(\,\cdot\,;t,x)\colon \mathscr{U}[t,T] \to \mathbb{R}$ given by

$$J(\boldsymbol{u}(\cdot);t,x) \doteq Y(t;t) \tag{1.8}$$

(see also Remark 1.2.7).

Remark 1.2.10. The functional $J(\cdot\,; t, x)$ in (1.8) is a real-valued generalized Bolza-type functional and, for any $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$,

$$J(\boldsymbol{u}(\cdot); t, x) = \mathbf{E}\left[\int_t^T f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t); t)\, ds + h(X(T); t)\right],$$

an expression in which the *running* or *intertemporal utility* and the *terminal utility* are explicitly specified (see again Remark 1.2.7). In particular, the more constant f is w.r.t. the variables y and z, the more we return to the classical sphere of stochastic optimal control theory.

Definition 1.2.7 (Equilibrium policy)**.** Suppose Assumptions 1 and 2 hold. We call a *(subgame-perfect) equilibrium policy associated with* T, I, U, W *and* b, σ, f, h any measurable map

$$\mathbf{\Pi}\colon [0,T]_s \times I_x \to U$$

such that, for any $t \in [0, T[$ and $x \in I$, there exists a unique I-valued Itô process

$$\bar{X}(\cdot) \coloneqq X^{t,x,\mathbf{\Pi}}(\cdot) \tag{1.9}$$

that is a solution of the FSDE

$$\begin{cases} dX(s) = b\big(s, X(s), \mathbf{\Pi}(s, X(s))\big)\, ds + \sigma\big(s, X(s), \mathbf{\Pi}(s, X(s))\big)\, dW(s), \\ X(t) = x, \end{cases}$$

(where $s \in [t,T]$) belonging to $\mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R})$ and is such that if we denote, for any $s \in [t,T]$ (and $\mathbf{P}$-a.s.),

$$\bar{\boldsymbol{u}}(s) \doteq \mathbf{\Pi}(s, \bar{X}(s)), \tag{1.10}$$

then we have $\bar{\boldsymbol{u}}(\cdot) \in \mathscr{U}[t,T]$ and, for any other $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$,

$$\liminf_{\varepsilon \downarrow 0} \frac{J(\bar{\boldsymbol{u}}^{\varepsilon}(\cdot); t, x) - J(\bar{\boldsymbol{u}}(\cdot); t, x)}{\varepsilon} \geq 0, \tag{1.11}$$

where, for $\varepsilon \downarrow 0$, $\bar{\boldsymbol{u}}^{\varepsilon}(\cdot)$ is the spike variation of $\bar{\boldsymbol{u}}(\cdot)$ w.r.t. $\boldsymbol{u}(\cdot)$ and E_t^{ε} given by

$$E_t^{\varepsilon} \coloneqq [t, t+\varepsilon] \tag{1.12}$$

(see also Definition 1.2.2).

Remark 1.2.11. The $\liminf_{\varepsilon\downarrow 0}$ in (1.11) will turn out to be an actual limit (see Lemma 2.1.1 in Chapter 2).

Notation 1.2.4. With respect to the notation of Definition 1.2.7, we specify the dependence on the elements involved by writing

$$\bar{X}(\cdot) \coloneqq X^{t,x,\bar{\boldsymbol{u}}}(\cdot), \quad \bar{Y}(\cdot\,;t) \coloneqq Y^{x,\bar{\boldsymbol{u}}}(\cdot\,;t), \quad \bar{Z}(\cdot\,;t) \coloneqq Z^{x,\bar{\boldsymbol{u}}}(\cdot\,;t)$$

and

$$X^{\varepsilon}(\cdot) \coloneqq X^{t,x,\bar{\boldsymbol{u}}^{\varepsilon}}(\cdot), \quad Y^{\varepsilon}(\cdot\,;t) \coloneqq Y^{x,\bar{\boldsymbol{u}}^{\varepsilon}}(\cdot\,;t), \quad Z^{\varepsilon}(\cdot\,;t) \coloneqq Z^{x,\bar{\boldsymbol{u}}^{\varepsilon}}(\cdot\,;t)$$

(similarly to Notations 1.2.2 and 1.2.3).

Definition 1.2.8 (Equilibrium control/pair/4-tuple)**.** With respect to the notations of Definition 1.2.7 and Notation 1.2.4, we call:

$$\bar{\boldsymbol{u}}(\cdot)$$

an *(subgame-perfect) equilibrium control* (or *strategy*),

$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot))$$

an *(subgame-perfect) equilibrium pair* and

$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$$

an *(subgame-perfect) equilibrium 4-tuple.*

Remark 1.2.12. Regarding Definitions 1.2.7 and 1.2.8, we point out the following.

- If $\bar{\boldsymbol{u}}(\cdot)$ as in (1.10) is an optimal control in the classical sense, i.e., $\bar{\boldsymbol{u}}(\cdot)$ minimizes the objective functional $J(\cdot\,;t,x)$ over $\mathscr{U}[t,T]$, then $\bar{\boldsymbol{u}}(\cdot)$ is also an equilibrium control (the opposite cannot be true, in general).
- We could specify an equilibrium policy/control/pair/4-tuple to be *strong* in cases in which the inequality in (1.11) is strong, i.e., narrow (adjusting the entire sequel accordingly).
- In general, we cannot expect an equilibrium policy/control/pair/4-tuple to be unique, even if it exists. It might therefore be an idea to select one uniquely through a constraint (as in, e.g., Definition 1.2.9).
- If $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot))$ is an equilibrium pair, then, by (1.10),

  $$\bar{\boldsymbol{u}}(t) = \boldsymbol{\Pi}(t,x)$$

  which, by definition of $\boldsymbol{\Pi}$, is a deterministic constant (vector in U).

- The condition (1.11) can be rewritten as

$$\liminf_{\varepsilon \downarrow 0} \frac{Y^{\varepsilon}(t;t) - \bar{Y}(t;t)}{\varepsilon} \geq 0 \tag{1.13}$$

(see Definition 1.2.6 and Notation 1.2.4).

Definition 1.2.9 (State constraint)**.** Fix other deterministic maps

$$\boldsymbol{f} \colon [0,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}} \times \mathbb{R}_y \times \mathbb{R}_z \times [0,T[_t \to \mathbb{R}$$

and

$$\boldsymbol{h} \colon \mathbb{R}_x \times [0,T[_t \to \mathbb{R}$$

such that the analogue of Assumption 1, with $\boldsymbol{f}$ in place of f and $\boldsymbol{h}$ in place of h, holds, together with b and σ. For $t \in [0,T[$ and $x \in I$, choose a *utility domain*

$$\boldsymbol{\Gamma_{t,x}} \subset \mathbb{R}$$

that is a non-empty closed interval and, for $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$, let $X(\cdot)$ be the admissible state process as in Definition 1.2.3. Consider the admissible recursive utility

$$\boldsymbol{Y}(\cdot\,;t) \in \mathcal{L}^2_{\mathbb{F}}(\Omega;\mathcal{C}([t,T];\mathbb{R}))$$

corresponding to the recursive utility system

$$\begin{cases} d\boldsymbol{Y}(s;t) = -\boldsymbol{f}(s,X(s),\boldsymbol{u}(s),\boldsymbol{Y}(s;t),\boldsymbol{Z}(s;t);t)ds + \boldsymbol{Z}(s;t)dW(s), \\ \boldsymbol{Y}(T;t) = \boldsymbol{h}(X(T);t), \end{cases} \tag{1.14}$$

(where $s \in [t,T]$) and the utility functional $\boldsymbol{J}(\cdot\,;t,x) \colon \mathscr{U}[t,T] \to \mathbb{R}$ given by

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{u}(\cdot);t,x) &\doteq \boldsymbol{Y}(t;t) \\ &= \mathbf{E}\left[\int_t^T \boldsymbol{f}(s,X(s),\boldsymbol{u}(s),\boldsymbol{Y}(s;t),\boldsymbol{Z}(s;t);t)\,ds + \boldsymbol{h}(X(T);t)\right] \end{aligned} \tag{1.15}$$

(see Definition 1.2.4 and annexes). We call a *(one-dimensional) state constraint for* $(\boldsymbol{u}(\cdot),X(\cdot))$ the requirement

$$\boldsymbol{J}(\boldsymbol{u}(\cdot);t,x) \in \boldsymbol{\Gamma_{t,x}}, \tag{1.16}$$

and, if this is fulfilled, we say that $(\boldsymbol{u}(\cdot),X(\cdot))$ *satisfies the state constraint.*

Remark 1.2.13. Regarding Definition 1.2.9, we point out the following.

- The constraint (1.16) is imposed on the utility functional $\boldsymbol{J}(\cdot\,;t,x)$ and not directly on the state process $X(\cdot)$. The slight terminological abuse consisting in naming (1.16) as done above is inspired by [45, Chap. 3, Sect. 6].

- The state constraint (1.16) concerns $(\boldsymbol{u}(\cdot), X(\cdot))$ and not the whole 4-tuple $(\boldsymbol{u}(\cdot), X(\cdot), Y(\cdot\,;t), Z(\cdot\,;t))$.

- The use of bold type in the notation is preferred because what we will see would work even in the case of a non-empty closed and convex subset $\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$ of a multidimensional Euclidean space (see Chapter 3).

The two stochastic control problems that we will deal with, for which it is essentially a matter of obtaining a (Pontryagin) *maximum principle*, can be stated as follows (see Definitions 1.2.8 and 1.2.9).

Problem 1 (unconstrained)**.** Find *necessary and sufficient conditions* for an equilibrium 4-tuple.

Problem 2 (constrained)**.** Find *necessary conditions* for an equilibrium pair that satisfies the state constraint.

Remark 1.2.14. Having thus established definitions, assumptions and purposes, we want to rigorously emphasize that our "generalized" optimization problem (Problem 1) results generally affected by time-inconsistency exactly because of the shape of the recursive utility $Y(\cdot\,;t)$ (see Definition 1.2.4)—and so of the utility functional $J(\cdot\,;t,x)$ (see Definition 1.2.6)—which indeed we interpret financially as having a structure of non-exponential time discounting and, therefore, a not constant (psychological) discount rate. That fact, on the other hand, explains why (subgame-perfect) equilibrium controls/strategies are considered (Definitions 1.2.7 and Definition 1.2.8). For all this, we also refer to the portfolio management problem of Section 2.4 in Chapter 2.

1.3 Preliminary results

We recall a standard estimate for BSDEs that is decisive in [22] (on which we rely) and that can be found in, e.g., [5]. Its meaning is essentially the continuous dependence of the pair solution on the assigned data. We will use it directly in Lemma 1.3.3 below (which we will need in Chapter 3).

Lemma 1.3.1. *Take measurable maps* $p \in\,]1,\infty[$, $\xi_t, \hat{\xi}_t \in L^p_T(\Omega;\mathbb{R})$, *and*

$$F, \hat{F} \colon [0,T]_s \times \mathbb{R}_y \times \mathbb{R}_z \times \Omega_\omega \times [0,T[_t \to \mathbb{R}$$

that are s-$\mathbf{P}$-uniformly Lipschitz w.r.t. (y,z) such that, for any $y, z \in \mathbb{R}$, $F(\cdot, y, z; t)$ and $\hat{F}(\cdot, y, z; t)$ are $\mathbb{R}$-valued $(\mathcal{F}_s)_{s\in[t,T]}$-progressively measurable processes with $F(\cdot, 0, 0; t)$, $\hat{F}(\cdot, 0, 0; t) \in \mathcal{L}^p_{\mathbb{F}}(t, T; \mathbb{R})$. Consider the BSDEs with parameters $(-F, \xi_t)$ and $(-\hat{F}, \hat{\xi}_t)$ respectively, i.e.,

$$\begin{cases} d\Xi(s;t) = -F(s, \Xi(s;t), \mathrm{O}(s;t); t)ds + \mathrm{O}(s;t)dW(s), \\ \Xi(T;t) = \xi_t, \end{cases}$$

and

$$\begin{cases} d\hat{\Xi}(s;t) = -\hat{F}(s, \hat{\Xi}(s;t), \hat{\mathrm{O}}(s;t); t)ds + \hat{\mathrm{O}}(s;t)dW(s), \\ \hat{\Xi}(T;t) = \hat{\xi}_t. \end{cases}$$

(where $s \in [t, T]$). Then there exists a constant $K_p \in]0, \infty[$ such that

$$\begin{aligned} \mathbf{E} \sup_{s\in[t,T]} \left|\Xi(s;t) - \hat{\Xi}(s;t)\right|^p + \mathbf{E}\left(\int_t^T \left|\mathrm{O}(s;t) - \hat{\mathrm{O}}(s;t)\right|^2 ds\right)^{p/2} &\leq K_p\, \mathbf{E} \left|\xi_t - \hat{\xi}_t\right|^p \\ + K_p\, \mathbf{E}\left(\int_t^T \left|F(s, \Xi(s;t), \mathrm{O}(s;t); t) - \hat{F}(s, \Xi(s;t), \mathrm{O}(s;t); t)\right| ds\right)^p&. \end{aligned}$$

Remark 1.3.1. Regarding Lemma 1.3.1, we point out the following.

- The constant K_p depends also on t, T, and the Lipschitz constants, but neither on $\xi_t(\cdot)$, $\hat{\xi}_t(\cdot)$, $\Xi(\cdot\,; t)$, $\mathrm{O}(\cdot\,; t)$ nor on $\hat{\Xi}(\cdot\,; t)$, $\hat{\mathrm{O}}(\cdot\,; t)$.
- The main result underlying the whole theory of BSDEs is the classic representation theorem of integrable square continuous martingales, and thus it is crucial that the reference filtration remains the completed filtration $\mathbb{F}$ generated by W.
- The two exponentiations to powers $p/2$ and p concern the two deterministic integrals (the expected values of which we then calculate), and not just the respective integral functions (which are absolute values of differences). See also Remark 1.1.2.

Definition 1.3.1 (metric $\mathbf{d}$ on $\mathscr{U}[t, T]$)**.** Let $\mathbf{d}$ be defined, for any $\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot) \in \mathscr{U}[t, T]$, by

$$\begin{aligned} \mathbf{d}(\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot)) &\doteq (\mathbf{m} \otimes \mathbf{P})[\{(s, \omega) \in [t, T] \times \Omega \mid \boldsymbol{u}(s, \omega) \neq \hat{\boldsymbol{u}}(s, \omega)\}] \\ &= \mathbf{E} \int_t^T \mathbb{1}_{\{\boldsymbol{u}(\cdot)\neq\hat{\boldsymbol{u}}(\cdot)\}}(s, \omega)ds. \end{aligned} \tag{1.17}$$

Remark 1.3.2. Regarding Definition 1.3.1, we point out the following.

- For any $\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot) \in \mathscr{U}[t,T]$,

$$\mathbf{d}(\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot)) \leq T - t.$$

- We are taking the quotient space of $\mathscr{U}[t,T]$ w.r.t. the following equivalence relation $\sim$: for any $\boldsymbol{u}(\cdot), \tilde{\boldsymbol{u}}(\cdot) \in \mathscr{U}[t,T]$, $\boldsymbol{u}(\cdot) \sim \tilde{\boldsymbol{u}}(\cdot)$ if and only if

$$\mathbf{d}(\boldsymbol{u}(\cdot), \tilde{\boldsymbol{u}}(\cdot)) = 0$$

  (a relation that is weaker, and therefore more restrictive, than that of indistinguishability).

The following result is essentially Lemma 6.4 in [45, Chap. 3, Sect. 6]. See also Remark 1.2.1.

Lemma 1.3.2. *Suppose that U is bounded. Then $(\mathscr{U}[t,T], \mathbf{d})$ is a complete metric space.*

Now, under Assumption 1, let us go back to consider the utility functionals J as in Definition 1.2.6 (see (1.8)) and $\boldsymbol{J}$ as in Definition 1.2.9 (see (1.15)), to prove the following continuity property.

Lemma 1.3.3. *Suppose that U is bounded and that Assumption 1 holds, and fix $t \in [0, T[$ and $x \in I$. Then the two utility functionals*

$$J(\,\cdot\,; t, x), \boldsymbol{J}(\,\cdot\,; t, x) \colon \mathscr{U}[t,T] \to \mathbb{R}$$

are continuous w.r.t. the metric $\mathbf{d}$.

Proof. It is enough to prove the claim for $J(\,\cdot\,; t, x)$. Fix $\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot) \in \mathscr{U}[t,T]$ and consider the corresponding admissible 4-tuples $(\boldsymbol{u}(\cdot), X(\cdot), Y(\,\cdot\,; t), Z(\,\cdot\,; t))$ and $(\hat{\boldsymbol{u}}(\cdot), \hat{X}(\cdot), \hat{Y}(\,\cdot\,; t), \hat{Z}(\,\cdot\,; t))$ (see Definition 1.2.5 and annexes).

So, by Lemma 1.3.1 and Jensen's inequality (also in its discrete version),

$$\begin{aligned}
&\left|J(\boldsymbol{u}(\cdot);t,x) - J(\hat{\boldsymbol{u}}(\cdot);t,x)\right|^p \\
&\equiv \left|Y(t;t) - \hat{Y}(t;t)\right|^p \\
&\lesssim \mathbf{E}\left|h(X(T);t) - h(\hat{X}(T);t)\right|^p \\
&\quad + \mathbf{E}\bigg(\int_t^T \Big| f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t);t) \\
&\qquad\qquad - f(s, \hat{X}(s), \hat{\boldsymbol{u}}(s), Y(s;t), Z(s;t);t)\Big| ds\bigg)^p \\
&\lesssim \mathbf{E}\left|X(T) - \hat{X}(T)\right|^p + \mathbf{E}\left(\int_t^T \left(\left|X(s) - \hat{X}(s)\right| + |\boldsymbol{u}(s) - \hat{\boldsymbol{u}}(s)|\right) ds\right)^p \\
&\lesssim \mathbf{E}\left|X(T) - \hat{X}(T)\right|^p + \mathbf{E}\int_t^T \left|X(s) - \hat{X}(s)\right|^p ds + \mathbf{E}\int_t^T |\boldsymbol{u}(s) - \hat{\boldsymbol{u}}(s)|^p ds \\
&\lesssim \mathbf{E}\sup_{s\in[t,T]}\left|X(s) - \hat{X}(s)\right|^p + \mathbf{d}(\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot))
\end{aligned}$$

(in particular, $(Y(\cdot\,;t), Z(\cdot\,;t))$ and $(\hat{Y}(\cdot\,;t), \hat{Z}(\cdot\,;t))$ have disappeared), where we have also used the boundedness of U for the term with $|\boldsymbol{u}(\cdot) - \hat{\boldsymbol{u}}(\cdot)|^p$ once we have noted that

$$\mathbf{E}\int_t^T |\boldsymbol{u}(s) - \hat{\boldsymbol{u}}(s)|^p ds = \mathbf{E}\int_t^T \mathbb{1}_{\{\boldsymbol{u}(\cdot)\neq\hat{\boldsymbol{u}}(\cdot)\}}(s,\omega)|\boldsymbol{u}(s) - \hat{\boldsymbol{u}}(s)|^p ds$$

(see (1.17) of Definition 1.3.1). Our claim now is that

$$\mathbf{E}\sup_{s\in[t,T]}\left|X(s) - \hat{X}(s)\right|^p \lesssim \mathbf{E}\int_t^T |\boldsymbol{u}(s) - \hat{\boldsymbol{u}}(s)|^p ds \tag{1.18}$$

($\lesssim \mathbf{d}(\boldsymbol{u}(\cdot), \hat{\boldsymbol{u}}(\cdot))$ again, and so on), which allows us to conclude the proof.

Indeed, for any $s \in [t,T]$,

$$\begin{aligned}
\left|X(s) - \hat{X}(s)\right| \le \int_t^s &\left|b(r, X(r), \boldsymbol{u}(r)) - b(r, \hat{X}(r), \hat{\boldsymbol{u}}(r))\right| dr \\
&+ \left|\int_t^s \big(\sigma(r, X(r), \boldsymbol{u}(r)) - \sigma(r, \hat{X}(r), \hat{\boldsymbol{u}}(r))\big)\, dW(r)\right|
\end{aligned}$$

(see (1.4)), from which, in a similar way to the above,

$$\left|X(s)-\hat{X}(s)\right|^p \lesssim \int_t^s \left|b(r,X(r),\boldsymbol{u}(r)) - b(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\right|^p dr + \left|\int_t^s \big(\sigma(r,X(r),\boldsymbol{u}(r)) - \sigma(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\big)\, dW(r)\right|^p.$$

Therefore, for any $\tau \in [t,T]$, by the classical Burkholder–Davis–Gundy inequality (with temporal variables in $[t,T]$ and with exponentiation $r := p/2$) and Jensen's inequality (since $p \geq 2$),

$$\begin{aligned}
&\mathbf{E} \sup_{s\in[t,\tau]} \left|X(s)-\hat{X}(s)\right|^p \\
&\lesssim \mathbf{E}\int_t^\tau \left|b(r,X(r),\boldsymbol{u}(r)) - b(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\right|^p dr \\
&\quad + \mathbf{E}\sup_{s\in[t,\tau]} \left|\int_t^s \big(\sigma(r,X(r),\boldsymbol{u}(r)) - \sigma(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\big)\, dW(r)\right|^p \\
&\lesssim \mathbf{E}\int_t^\tau \left|b(r,X(r),\boldsymbol{u}(r)) - b(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\right|^p dr \\
&\quad + \mathbf{E}\left(\int_t^\tau \big(\sigma(r,X(r),\boldsymbol{u}(r)) - \sigma(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\big)^2 dr\right)^{p/2} \\
&\leq \mathbf{E}\int_t^\tau \Big(\left|b(r,X(r),\boldsymbol{u}(r)) - b(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\right|^p \\
&\qquad\qquad + \left|\sigma(r,X(r),\boldsymbol{u}(r)) - \sigma(r,\hat{X}(r),\hat{\boldsymbol{u}}(r))\right|^p\Big)\, dr \\
&\lesssim \mathbf{E}\int_t^\tau \left|X(r)-\hat{X}(r)\right|^p dr + \mathbf{E}\int_t^\tau |\boldsymbol{u}(r)-\hat{\boldsymbol{u}}(r)|^p\, dr \\
&\leq \int_t^\tau \mathbf{E}\sup_{s\in[t,r]} \left|X(s)-\hat{X}(s)\right|^p dr + \mathbf{E}\int_t^\tau |\boldsymbol{u}(r)-\hat{\boldsymbol{u}}(r)|^p\, dr
\end{aligned}$$

and thus, by Grönwall's inequality (in its integral form) applied to the continuous function

$$\tau \mapsto \mathbf{E}\sup_{s\in[t,\tau]} \left|X(s)-\hat{X}(s)\right|^p$$

(on $[t,T]$), we get

$$\mathbf{E}\sup_{s\in[t,\tau]} \left|X(s)-\hat{X}(s)\right|^p \lesssim \mathbf{E}\int_t^\tau |\boldsymbol{u}(r)-\hat{\boldsymbol{u}}(r)|^p\, dr,$$

hence what we wanted, namely, (1.18), by choosing $\tau = T$. □

Remark 1.3.3. Regarding this proof of Lemma 1.3.3, we could have done the calculations by using

$$p = 2$$

(and explicitly that, for $a_1, a_2 \in \mathbb{R}$, $(a_1 + a_2)^2 \le 2(a_1^2 + a_2^2)$). Furthermore, we can see that

$$\hat{\boldsymbol{u}}(\cdot) \xrightarrow{\mathbf{d}} \boldsymbol{u}(\cdot) \qquad \Longrightarrow \qquad \begin{cases} \hat{Y}(t;t) \xrightarrow{\mathbb{R}} Y(t;t) \\ \hat{X}(T) \xrightarrow{L^2} X(T) \end{cases}$$

(see (1.18)).

We conclude the current chapter with a brief discussion of the classic comparison theorem for BSDEs, which, in the context of linearity, boils down to a simple observation, Remark 1.3.4 below, which will be important for our discussion, especially because what we will call *adjoint equations* will be linear BSDEs. See, e.g., [17], where it is also possible to extrapolate the following starting result.

Proposition 1.3.1. *Fix $t \in [0, T[$. Take*

$$\beta(\cdot\,; t), \gamma(\cdot\,; t) \in \mathcal{L}^{\infty}_{\mathbb{F}}(t, T; \mathbb{R})$$

and $\eta(\cdot\,; t)$ such that

$$\begin{cases} d\eta(s;t) = \eta(s;t)\big[\beta(s;t)ds + \gamma(s;t)dW(s)\big], \\ \eta(t;t) = 1, \end{cases}$$

(where $s \in [t, T]$) i.e., explicitly,

$$\eta(s;t) = \exp\left\{\int_t^s \left[\beta(r;t) - \frac{\gamma^2(r;t)}{2}\right] dr + \int_t^s \gamma(r;t)\, dW(r)\right\}.$$

Then, for any $\alpha(\cdot\,; t) \in \mathcal{L}^2_{\mathbb{F}}(t, T; \mathbb{R})$ and $\xi_t \in L^2_T(\Omega; \mathbb{R})$, there exists a unique pair solution

$$(\Xi(\cdot\,; t), \mathrm{O}(\cdot\,; t)) \in \mathcal{L}^2_{\mathbb{F}}(\Omega; \mathcal{C}([t, T]; \mathbb{R})) \times \mathcal{L}^2_{\mathbb{F}}(t, T; \mathbb{R})$$

of the BSDE

$$\begin{cases} d\Xi(s;t) = -\big[\alpha(s;t) + \beta(s;t)\Xi(s;t) + \gamma(s;t)\mathrm{O}(s;t)\big]ds + \mathrm{O}(s;t)dW(s), \\ \Xi(T;t) = \xi_t, \end{cases} \tag{1.19}$$

(where $s \in [t, T]$) and the process $\Xi(\cdot\,; t)$ is the conditional expectation given by, for any $s \in [t, T]$ (and $\mathbf{P}$-a.s.),

$$\Xi(s;t) = \eta^{-1}(s;t)\, \mathbf{E}\left[\eta(T;t)\xi_t + \int_s^T \eta(r;t)\alpha(r;t)\, dr \,\middle|\, \mathcal{F}_s\right]. \tag{1.20}$$

Remark 1.3.4. Regarding Proposition 1.3.1, we point out the following.

- Since $\eta(\cdot\,;t) > 0$, it follows from (1.20) that

$$\begin{cases} \xi_t \geq 0 \\ \alpha(\cdot\,;t) \geq 0 \end{cases} \qquad \Longrightarrow \qquad \Xi(\cdot\,;t) \geq 0 \tag{1.21}$$

  (and similarly with $\leq$ everywhere), and, moreover, the narrow inequality for $\Xi(\cdot\,;t)$ holds even if only one of the two other inequalities is narrow: e.g.,

$$\begin{cases} \xi_t > 0 \\ \alpha(\cdot\,;t) \geq 0 \end{cases} \qquad \Longrightarrow \qquad \Xi(\cdot\,;t) > 0.$$

- In general, for $\tau \in [s, T]$,

$$\eta(\tau;t)\,\eta^{-1}(s;t) \neq \eta(\tau;s)$$

  (owing to the dependence on t of $\beta(\cdot\,;t)$ and $\gamma(\cdot\,;t)$), and therefore we can expect that, as processes,

$$\Xi(s;t) \neq \mathbf{E}\left[\eta(T;s)\xi_t + \int_s^T \eta(r;s)\alpha(r;t)\,dr \,\middle|\, \mathcal{F}_s\right]$$

  (in which the latter term differs from $\Xi(s;s)$ through the dependence on t of $\alpha(\cdot\,;t)$ and ξ_t).

Chapter 2

Unconstrained problem: necessary and sufficient conditions

In this chapter, we solve Problem 1 by adapting the calculations of [22] appropriately. In particular, heuristics relating to the shape of the adjoint equations/processes and their respective generalized Hamiltonian functions are not provided. We suppose that Assumptions 1 and 2 hold, and we fix ($t \in [0, T[$, $x \in I$ and) an admissible 4-tuple

$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,; t), \bar{Z}(\cdot\,; t))$$

(see Definition 1.2.5) that we see as a candidate equilibrium 4-tuple (see Definitions 1.2.7 and 1.2.8).

2.1 A maximum principle

Notation 2.1.1. For any map $\varphi(s, x, \boldsymbol{u})$ between $b(s, x, \boldsymbol{u})$, $\sigma(s, x, \boldsymbol{u})$ and their derivatives up to second order, and for $s \in [t, T]$ (and $\mathbf{P}$-a.s.), we write

$$\varphi(s) \coloneqq \varphi(s, \bar{X}(s), \bar{\boldsymbol{u}}(s))$$

and, for $\boldsymbol{u}(\cdot) \in \mathscr{U}[t, T]$,

$$\delta\varphi(s) \coloneqq \varphi(s, \bar{X}(s), \boldsymbol{u}(s)) - \varphi(s),$$

while similarly, for any map $\varphi(s, x, \boldsymbol{u}, y, z; t)$ between $f(s, x, \boldsymbol{u}, y, z; t)$ and its derivatives up to second order, and for $s \in [t, T]$ (and $\mathbf{P}$-a.s.), we write

$$\varphi(s; t) \coloneqq \varphi(s, \bar{X}(s), \bar{\boldsymbol{u}}(s), \bar{Y}(s; t), \bar{Z}(s; t); t).$$

Regarding the following, let us keep in mind Notation 2.1.1.

Definition 2.1.1 ($\kappa(\cdot\,;t)$)**.** Associated with $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$, we define the process $\kappa(\cdot\,;t)$ as the solution of the linear FSDE

$$\begin{cases} d\kappa(s;t) = \kappa(s;t)\big[f_y(s;t)ds + f_z(s;t)dW(s)\big], \\ \kappa(t;t) = 1, \end{cases} \tag{2.1}$$

(where $s \in [t,T]$).

Remark 2.1.1. The process $\kappa(\cdot\,;t)$ in (2.1) is strictly positive and can be interpreted as a change of numéraire relative to the (dis)utility and corresponding to the coefficients $f_y(\cdot\,;t)$ and $f_z(\cdot\,;t)$.

Definition 2.1.2 (Adjoint equation/process of first order)**.** We call the *adjoint equation of first order associated with* $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ the linear BSDE

$$\begin{cases} dp(s;t) = -g(s,p(s;t),q(s;t);t)ds + q(s;t)dW(s), \\ p(T;t) = h_x(\bar{X}(T);t), \end{cases} \tag{2.2}$$

(where $s \in [t,T]$ and) where the map

$$g\colon [0,T]_s \times \mathbb{R}_p \times \mathbb{R}_q \times [0,T[_t \to \mathbb{R}$$

is given, for $s \in [t,T]$ (and $\mathbf{P}$-a.s.), by

$$g(s,p,q;t) \coloneqq \big[b_x(s) + f_z(s;t)\sigma_x(s) + f_y(s;t)\big]p \\ + \big[\sigma_x(s) + f_z(s;t)\big]q + f_x(s;t). \tag{2.3}$$

(the dependence on $\omega \in \Omega$ is implicit). We call *adjoint process of first order associated with* $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ any process

$$p(\cdot\,;t)$$

such that $(p(\cdot\,;t), q(\cdot\,;t))$ is a pair solution of (2.2) that belongs to

$$\mathcal{L}^2_{\mathbb{F}}(\Omega; \mathcal{C}([t,T];\mathbb{R})) \times \mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R}).$$

Remark 2.1.2. The process $p(\cdot\,;t)$, or the independent variable p of g in (2.3) are not to be confused with the summability exponent p (see Definition 1.2.1), especially because we might want the latter to take the fixed value $p = 2$.

Definition 2.1.3 (Adjoint equation/process of second order)**.** We call the *adjoint equation of second order associated with* $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ the linear BSDE

$$\begin{cases} dP(s;t) = -G(s,P(s;t),Q(s;t);t)ds + Q(s;t)dW(s), \\ P(T;t) = h_{xx}(\bar{X}(T);t), \end{cases} \tag{2.4}$$

(where $s \in [t,T]$ and) where the map

$$G\colon [0,T]_s \times \mathbb{R}_P \times \mathbb{R}_Q \times [0,T[_t \to \mathbb{R}$$

is given, for $s \in [t,T]$ (and **P**-a.s.), by

$$\begin{aligned} G(s,P,Q;t) &\coloneqq \big[2b_x(s) + \sigma_x(s)^2 + 2f_z(s;t)\sigma_x(s) + f_y(s;t)\big]P \\ &+ \big[2\sigma_x(s) + f_z(s;t)\big]Q + b_{xx}(s)p(s;t) + \sigma_{xx}(s)\big[f_z(s;t)p(s;t) + q(s;t)\big] \\ &\qquad + \big(1, p(s;t), \sigma_x(s)p(s;t) + q(s;t)\big) \\ &\qquad\qquad \cdot \mathrm{D}^2 f(s;t) \\ &\qquad\qquad\quad \cdot \big(1, p(s;t), \sigma_x(s)p(s;t) + q(s;t)\big)^{\mathsf{T}}. \end{aligned} \tag{2.5}$$

(the dependence on $\omega \in \Omega$ is implicit). We call *adjoint process of second order associated with* $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ any process

$$P(\cdot\,;t)$$

such that $(P(\cdot\,;t), Q(\cdot\,;t))$ is a pair solution of (2.4) that belongs to

$$\mathcal{L}^2_{\mathbb{F}}(\Omega;\mathcal{C}([t,T];\mathbb{R})) \times \mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R}).$$

For the adjoint equations (2.2) and (2.4) of Definitions 2.1.2 and 2.1.3, respectively, the following result regarding existence, uniqueness, and regularity holds (see [22]).

Proposition 2.1.1. *There exist unique pair solutions* $(p(\cdot\,;t), q(\cdot\,;t))$ *of* (2.2) *and* $(P(\cdot\,;t), Q(\cdot\,;t))$ *of* (2.4) *such that, for any* $k \in [1,\infty[$,

$$\mathbf{E}\sup_{s\in[t,T]}\Big[|p(s;t)|^{2k} + |P(s;t)|^{2k}\Big] + \mathbf{E}\bigg(\int_t^T \Big[|q(s;t)|^2 + |Q(s;t)|^2\Big]\,ds\bigg)^k < \infty.$$

Remark 2.1.3. $p(t;t)$ and $P(t;t)$ are deterministic constants (for similar reasons to those given in Remark 1.2.7), while it is not possible to say the same in general about $q(t;t)$ and $Q(t;t)$.

Definition 2.1.4 (Generalized Hamiltonian function of first order)**.** We call the *generalized Hamiltonian (function) of first order associated with* b,σ,f the map

$$H\colon [0,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}} \times \mathbb{R}_y \times \mathbb{R}_z \times \mathbb{R}_p \times \mathbb{R}_q \times [0,T[_t \times I_{\bar{x}} \times U_{\bar{\boldsymbol{u}}} \to \mathbb{R}$$

given by

$$\begin{aligned} H(s,x,\boldsymbol{u},y,z,p,q;t,\bar{x},\bar{\boldsymbol{u}}) &\doteq pb(s,x,\boldsymbol{u}) + q\sigma(s,x,\boldsymbol{u}) \\ &\quad + f(s,x,\boldsymbol{u},y,z + p[\sigma(s,x,\boldsymbol{u}) - \sigma(s,\bar{x},\bar{\boldsymbol{u}})];t). \end{aligned} \tag{2.6}$$

Definition 2.1.5 (Generalized Hamiltonian function of second order)**.** We call the *generalized Hamiltonian (function) of second order associated with* b, σ, f the map

$$\mathcal{H}\colon [0,T]_s \times \mathbb{R}_x \times U_{\boldsymbol{u}} \times \mathbb{R}_y \times \mathbb{R}_z \times \mathbb{R}_p \times \mathbb{R}_q \times \mathbb{R}_P \times [0,T[_t \times I_{\bar{x}} \times U_{\bar{\boldsymbol{u}}} \to \mathbb{R}$$

given by

$$\begin{aligned}\mathcal{H}(s,x,\boldsymbol{u},y,z,p,q,P;t,\bar{x},\bar{\boldsymbol{u}}) &\doteq H(s,x,\boldsymbol{u},y,z,p,q;t,\bar{x},\bar{\boldsymbol{u}}) \\ &\quad + \tfrac{1}{2}P[\sigma(s,x,\boldsymbol{u}) - \sigma(s,\bar{x},\bar{\boldsymbol{u}})]^2. \end{aligned} \tag{2.7}$$

Notation 2.1.2. With respect to $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ and the corresponding adjoint processes $p(\cdot\,;t)$ and $P(\cdot\,;t)$ as in Definitions 2.1.2 and 2.1.3 respectively, and for $s \in [t,T]$ (and $\mathbf{P}$-a.s.), we write

$$\begin{aligned}&\mathcal{H}(s;t) \\ &:= \mathcal{H}(s,\bar{X}(s),\bar{\boldsymbol{u}}(s),\bar{Y}(s;t),\bar{Z}(s;t),p(s;t),q(s;t),P(s;t);t,\bar{X}(s),\bar{\boldsymbol{u}}(s))\end{aligned}$$

and, for $\boldsymbol{u} \in U$,

$$\begin{aligned}&\delta\mathcal{H}(s;t,\boldsymbol{u}) \\ &:= \mathcal{H}(s,\bar{X}(s),\boldsymbol{u},\bar{Y}(s;t),\bar{Z}(s;t),p(s;t),q(s;t),P(s;t);t,\bar{X}(s),\bar{\boldsymbol{u}}(s)) - \mathcal{H}(s;t)\end{aligned}$$

Regarding the following, let us keep in mind Notation 2.1.2.

Remark 2.1.4. For $s \in [t,T]$ and $\boldsymbol{u} \in U$, $\mathcal{H}(s;t)$ and $\delta\mathcal{H}(s;t,\boldsymbol{u})$ belong to $L^1_s(\Omega;\mathbb{R})$ and, moreover,

$$\delta\mathcal{H}(s;t,\bar{\boldsymbol{u}}(s)) = 0. \tag{2.8}$$

The key result is the following lemma, which we will prove in Section 2.2.

Lemma 2.1.1. *Fix* $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$. *For* $\varepsilon \in\,]0, T-t[$ *and* $s \in [t,T]$, *let* $E^\varepsilon_t \equiv E^\varepsilon_{t,s}$ *be given by*

$$E^\varepsilon_t := \begin{cases} [s, s+\varepsilon], & \text{if } s < T, \\ [T-\varepsilon, T], & \text{if } s = T, \end{cases} \tag{2.9}$$

and let $\bar{\boldsymbol{u}}^\varepsilon(\cdot)$ *be the spike variation of* $\bar{\boldsymbol{u}}(\cdot)$ *w.r.t.* $\boldsymbol{u}(\cdot)$ *and* E^ε_t. *Then, for any* $s \in [t,T]$, *the* $\liminf_{\varepsilon\downarrow 0}$ *as in* (1.11) *is an actual limit and takes the form*

$$\liminf_{\varepsilon\downarrow 0} \frac{J(\bar{\boldsymbol{u}}^\varepsilon(\cdot);t,x) - J(\bar{\boldsymbol{u}}(\cdot);t,x)}{\varepsilon} = \mathbf{E}\big[\kappa(s;t)\delta\mathcal{H}(s;t,\boldsymbol{u}(s))\big]. \tag{2.10}$$

Remark 2.1.5. As we will understand shortly, instead of (2.9), we could take, among other possibilities,

$$E_t^\varepsilon := \begin{cases} [s, s+\varepsilon[, & \text{if } s < T, \\ ]T-\varepsilon, T], & \text{if } s = T. \end{cases}$$

Corollary 2.1.1 (Sufficient conditions)**.** *Suppose there exists a measurable map* $\mathbf{\Pi}\colon [0,T] \times I \to U$ *such that, for any* $s \in [t, T]$ *(and* $\mathbf{P}$*-a.s.),*

$$\bar{\boldsymbol{u}}(s) = \mathbf{\Pi}(s, \bar{X}(s)) \tag{2.11}$$

and suppose that, for any $\boldsymbol{u} \in U$ *(and* $\mathbf{P}$*-a.s.),*

$$\delta\mathcal{H}(t; t, \boldsymbol{u}) \geq 0. \tag{2.12}$$

Then $\mathbf{\Pi}$ *is an equilibrium policy, i.e.,*

$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,; t), \bar{Z}(\cdot\,; t))$$

is an equilibrium 4-tuple.

Proof. By (2.10) of Lemma 2.1.1, with E_t^ε as in (1.12), that is,

$$s = t,$$

the inequality (1.11) holds (see also Definition 2.1.1). □

Remark 2.1.6. Regarding Corollary 2.1.1, we point out that, *if* $\bar{Z}(t; t)$ and $q(t; t)$ are deterministic constants, then the condition (2.12) is equivalent to

$$\mathbf{\Pi}(t, x) \in \operatorname*{arg\,min}_{\boldsymbol{u} \in U} \mathcal{H}(t, x, \boldsymbol{u}, \bar{Y}(t; t), \bar{Z}(t; t), p(t; t), q(t; t), P(t; t); t, x, \mathbf{\Pi}(t, x)) \tag{2.13}$$

because, by (2.11),

$$\mathbf{\Pi}(t, x) = \bar{\boldsymbol{u}}(t).$$

We are finally ready to present the first of our main results (see also Corollary 2.1.1 and Remark 2.1.6).

Theorem 2.1.1 (Maximum principle)**.** *Suppose there exists a measurable map* $\mathbf{\Pi}\colon [0,T] \times I \to U$ *such that, for any* $s \in [t, T]$ *(and* $\mathbf{P}$*-a.s.),*

$$\bar{\boldsymbol{u}}(s) = \mathbf{\Pi}(s, \bar{X}(s)).$$

Then the following three conditions are equivalent.

1. $\mathbf{\Pi}$ *is an equilibrium policy, i.e.,* $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,; t), \bar{Z}(\cdot\,; t))$ *is an equilibrium 4-tuple.*

2. *For any* $\boldsymbol{u} \in U$ *(and* $\mathbf{P}$*-a.s.),*

$$\delta\mathcal{H}(t; t, \boldsymbol{u}) \geq 0.$$

3. *If* $\bar{Z}(t; t)$ *and* $q(t; t)$ *are deterministic constants, then*

$$\mathbf{\Pi}(t, x) \in \arg\min_{\boldsymbol{u} \in U} \mathcal{H}(t, x, \boldsymbol{u}, \bar{Y}(t; t), \bar{Z}(t; t), p(t; t), q(t; t), P(t; t); t, x, \mathbf{\Pi}(t, x)).$$

Proof. In light of what we saw in Corollary 2.1.1 and Remark 2.1.6, we just need to show that

$$\mathbf{1} \Rightarrow \mathbf{2}$$

(necessary conditions, we would now say). To this end suppose, by contradiction, that there exist $t^* \in [0, T[$, $\boldsymbol{u}^* \in U$, and $\mathcal{N} \in \mathcal{F}$ with $\mathbf{P}[\mathcal{N}] > 0$, such that

$$\delta\mathcal{H}(t^*; t^*, \boldsymbol{u}^*) < 0$$

on $\mathcal{N}$. Then any $\boldsymbol{u}(\cdot) \in \mathscr{U}[t^*, T]$ such that, $\mathbf{P}$-a.s.,

$$\boldsymbol{u}(t^*) = \begin{cases} \bar{\boldsymbol{u}}(t^*) & \text{on } \Omega \setminus \mathcal{N}, \\ \boldsymbol{u}^* & \text{on } \mathcal{N} \end{cases}$$

(e.g., the trivial one) satisfies

$$\delta\mathcal{H}(t^*; t^*, \boldsymbol{u}(t^*)) = \delta\mathcal{H}(t^*; t^*, \boldsymbol{u}^*)\, \mathbb{1}_{\mathcal{N}}$$

(see also (2.8)), and therefore, since $\mathbf{P}[\mathcal{N}] > 0$,

$$\mathbf{E}\big[\delta\mathcal{H}(t^*; t^*, \boldsymbol{u}(t^*))\big] < 0,$$

which is a contradiction (see also (2.10)). $\square$

Under appropriate assumptions on our coefficients, we can replace $\mathcal{H}$ with H in Theorem 2.1.1 thus obtaining the following result, which will be used in Section 2.4. Let us keep in mind also Definition 2.1.3.

Corollary 2.1.2. *Suppose there exists a measurable map* $\mathbf{\Pi}\colon [0, T] \times I \to U$ *such that, for any* $s \in [t, T]$ *(and* $\mathbf{P}$*-a.s.),*

$$\bar{\boldsymbol{u}}(s) = \mathbf{\Pi}(s, \bar{X}(s))$$

and suppose that $\bar{Z}(t; t)$ *and* $q(t; t)$ *are deterministic constants. If* $h(\cdot\,; t)$ *is convex and* $G(\cdot\,, 0, 0; t) \geq 0$*, then the following two conditions are equivalent.*

1. *$\mathbf{\Pi}$ is an equilibrium policy, i.e., $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,; t), \bar{Z}(\cdot\,; t))$ is an equilibrium 4-tuple.*

2. *$\mathbf{\Pi}(t, x) \in \arg\min_{\boldsymbol{u}\in U} H(t, x, \boldsymbol{u}, \bar{Y}(t; t), \bar{Z}(t; t), p(t; t), q(t; t); t, x, \mathbf{\Pi}(t, x))$.*

Proof. This is a direct consequence of Theorem 2.1.1: indeed, comparing ((2.4) and) (2.5) of Definition 2.1.3 with (1.19) of Proposition 1.3.1 (Section 1.3 of Chapter 1), we deduce that

$$P(\cdot\,; t) \geq 0$$

by (1.21) of Remark 1.3.4 (where $\xi_t \equiv h_{xx}(\bar{X}(T); t)$, $\alpha(\cdot\,; t) \equiv G(\cdot\,, 0, 0; t)$ and $\Xi(\cdot\,; t) \equiv P(\cdot\,; t)$) and so the term of $\mathcal{H}$ that depends on it, namely,

$$\tfrac{1}{2} P(t; t)[\sigma(t, x, \boldsymbol{u}) - \sigma(t, \bar{x}, \bar{\boldsymbol{u}}(t))]^2$$

is superfluous in calculating the minimum (as in (2.13)). □

2.2 A proof of Lemma 2.1.1

We start from the following notational convention, borrowed from [22], with regard to which there should be no misunderstandings in this section.

Notation 2.2.1. For $\varepsilon \downarrow 0$ and $\Lambda^\varepsilon(\cdot\,; t) = (\Lambda^\varepsilon(s; t))_{s\in[t,T]} \in \mathcal{L}^2_{\mathbb{F}}(t, T; \mathbb{R})$ (possibly a random variable), if, for any $k \in [1, \infty[$,

$$\mathbf{E}\left(\int_t^T |\Lambda^\varepsilon(s; t)|\, ds\right)^{2k} = o_{\varepsilon\downarrow 0}\big(\varepsilon^{2k}\big),$$

then we simply write

$$\Lambda^\varepsilon(\cdot\,; t) = o_{\varepsilon\downarrow 0}\big(\varepsilon\big).$$

Fix $\boldsymbol{u}(\cdot) \in \mathscr{U}[t, T]$. For $\varepsilon \in\,]0, T - t[$ and $s \in [t, T]$, let $E^\varepsilon_t \equiv E^\varepsilon_{t,s}$ be as in (2.9) (see Lemma 2.1.1) and let $\bar{\boldsymbol{u}}^\varepsilon(\cdot)$ be the spike variation of $\bar{\boldsymbol{u}}(\cdot)$ w.r.t. $\boldsymbol{u}(\cdot)$ and E^ε_t. Consider the usual approximate variational systems/processes of first and second order of the state process $\bar{X}(\cdot)$ w.r.t. the control perturbation $\bar{\boldsymbol{u}}^\varepsilon(\cdot)$: i.e., respectively,

$$\begin{cases} dX^\varepsilon_1(s; t) = b_x(s) X^\varepsilon_1(s; t)\, ds + \big[\sigma_x(s) X^\varepsilon_1(s; t) + \delta\sigma(s) \mathbb{1}_{E^\varepsilon_t}(s)\big]\, dW(s), \\ X^\varepsilon_1(t; t) = 0, \end{cases}$$

and

$$\begin{cases} dX^\varepsilon_2(s; t) = \big[b_x(s) X^\varepsilon_2(s; t) + \delta b(s) \mathbb{1}_{E^\varepsilon_t}(s) + \tfrac{1}{2} b_{xx}(s) X^\varepsilon_1(s; t)^2\big]\, ds \\ \qquad\qquad + \big[\sigma_x(s) X^\varepsilon_2(s; t) + \delta\sigma_x(s) X^\varepsilon_1(s; t) \mathbb{1}_{E^\varepsilon_t}(s) \\ \qquad\qquad\quad + \tfrac{1}{2} \sigma_{xx}(s) X^\varepsilon_1(s; t)^2\big]\, dW(s), \\ X^\varepsilon_2(t; t) = 0, \end{cases}$$

(where $s \in [t, T]$). The following result is Theorem 4.4 in [45, Chap. 3, Sect. 4] about "Taylor expansions."

Lemma 2.2.1. *For any $k \in [1, \infty[$,*

$$\sup_{s\in[t,T]} \mathbf{E}\Big[\big|X^\varepsilon(s) - \bar{X}(s)\big|^{2k}\Big] = O\big(\varepsilon^k\big),$$

$$\sup_{s\in[t,T]} \mathbf{E}\Big[|X_1^\varepsilon(s;t)|^{2k}\Big] = O\big(\varepsilon^k\big),$$

$$\sup_{s\in[t,T]} \mathbf{E}\Big[\big|X^\varepsilon(s) - \bar{X}(s) - X_1^\varepsilon(s;t)\big|^{2k}\Big] = O\big(\varepsilon^{2k}\big),$$

$$\sup_{s\in[t,T]} \mathbf{E}\Big[|X_2^\varepsilon(s;t)|^{2k}\Big] = O\big(\varepsilon^{2k}\big),$$

$$\sup_{s\in[t,T]} \mathbf{E}\Big[\big|X^\varepsilon(s) - \bar{X}(s) - X_1^\varepsilon(s;t) - X_2^\varepsilon(s;t)\big|^{2k}\Big] = o_{\varepsilon\downarrow 0}\big(\varepsilon^{2k}\big)$$

(see Notation 1.2.4). Furthermore,

$$h(X^\varepsilon(T);t) - h(\bar{X}(T);t) - h_x(\bar{X}(T);t)\big[X_1^\varepsilon(T;t) + X_2^\varepsilon(T;t)\big] \\ - \tfrac{1}{2} h_{xx}(\bar{X}(T);t) X_1^\varepsilon(T;t)^2 = o_{\varepsilon\downarrow 0}\big(\varepsilon\big) \quad (2.14)$$

(see Notation 2.2.1).

Remark 2.2.1. Regarding Lemma 2.2.1, we point out that (2.14) can be rewritten as

$$Y^\varepsilon(T;t) - \bar{Y}(T;t) = o_{\varepsilon\downarrow 0}\big(\varepsilon\big) + p(T;t)\big[X_1^\varepsilon(T;t) + X_2^\varepsilon(T;t)\big] + \tfrac{1}{2} P(T;t) X_1^\varepsilon(T;t)^2$$

(see Definitions 1.2.4, 2.1.2, and 2.1.3).

What we need to properly estimate is, for $\varepsilon \downarrow 0$,

$$Y^\varepsilon(t;t) - \bar{Y}(t;t)$$

(see (1.13) in Remark 1.2.12), while, in the sense of Lemma 2.2.1, we know something useful only about

$$Y^\varepsilon(T;t) - \bar{Y}(T;t)$$

(see Remark 2.2.1). Therefore, the idea is to reconstruct information by going back from this term by means of appropriate BSDEs (using the adjoint processes as in Definitions 2.1.2 and 2.1.3).

We recall below a basic calculation we use, namely, Itô integration by parts for a regular product.

Remark 2.2.2. Take $a(\cdot\,;t),\vartheta(\cdot\,;t),\lambda(\cdot\,;t),A(\cdot\,;t),\Theta(\cdot\,;t),\Lambda(\cdot\,;t)\in\mathcal{L}^2_{\mathbb{F}}(t,T;\mathbb{R})$ such that, on $[t,T]\times\Omega$,

$$\begin{aligned} d\lambda(s;t) &= a(s;t)ds+\vartheta(s;t)dW(s),\\ d\Lambda(s;t) &= A(s;t)ds+\Theta(s;t)dW(s). \end{aligned}$$

Then $d\langle\lambda(\cdot\,;t),\Lambda(\cdot\,;t)\rangle(s)=\vartheta(s;t)\Theta(s;t)ds$ and

$$\begin{aligned} d\big(\lambda(\cdot\,;t)\Lambda(\cdot\,;t)\big)(s) &= \lambda(s;t)d\Lambda(s;t)+d\lambda(s;t)\Lambda(s;t)+d\langle\lambda(\cdot\,;t),\Lambda(\cdot\,;t)\rangle(s)\\ &= \big[\lambda(s;t)A(s;t)+a(s;t)\Lambda(s;t)+\vartheta(s;t)\Theta(s;t)\big]ds\\ &\quad+\big[\lambda(s;t)\Theta(s;t)+\vartheta(s;t)\Lambda(s;t)\big]dW(s) \end{aligned}$$

(on $[t,T]\times\Omega$). Consequently,

$$d\big(\Lambda^2(\cdot\,;t)\big)(s)=2\big[A(s;t)\Lambda(s;t)+\tfrac12\Theta^2(s;t)\big]ds+2\Theta(s;t)\Lambda(s;t)dW(s)$$

and

$$\begin{aligned} &\mathbf{E}\big[\lambda(s;t)\Lambda(s;t)\big]-\mathbf{E}\big[\lambda(t;t)\Lambda(t;t)\big]\\ &=\mathbf{E}\bigg[\int_t^s\big[\lambda(r;t)A(r;t)+a(r;t)\Lambda(r;t)+\vartheta(r;t)\Theta(r;t)\big]\,dr\bigg]. \end{aligned}$$

Grouping w.r.t. $\mathbb{1}_{E^\varepsilon_t}(\cdot)$, $X^\varepsilon_1(\cdot\,;t)$, $X^\varepsilon_2(\cdot\,;t)$, $X^\varepsilon_1(\cdot\,;t)^2$ and $X^\varepsilon_1(\cdot\,;t)\mathbb{1}_{E^\varepsilon_t}(\cdot)$ (on $[t,T]\times\Omega$), we have

$$\begin{aligned} &d\Big(p(\cdot\,;t)\big[X^\varepsilon_1(\cdot\,;t)+X^\varepsilon_2(\cdot\,;t)\big]+\tfrac12P(\cdot\,;t)X^\varepsilon_1(\cdot\,;t)^2\Big)(s)\\ &=\bigg\{\big[p(s;t)\delta b(s)+q(s;t)\delta\sigma(s)+\tfrac12P(s;t)(\delta\sigma(s))^2\big]\mathbb{1}_{E^\varepsilon_t}(s)\\ &\qquad+\big[p(s;t)b_x(s)+q(s;t)\sigma_x(s)-g(s,p(s;t),q(s;t);t)\big]\big[X^\varepsilon_1(s;t)+X^\varepsilon_2(s;t)\big]\\ &\qquad+\tfrac12\Big[p(s;t)b_{xx}(s)+q(s;t)\sigma_{xx}(s)+2P(s;t)b_x(s)+2Q(s;t)\sigma_x(s)\\ &\qquad\qquad+P(s;t)\sigma_x(s)^2-G(s,P(s;t),Q(s;t);t)\Big]X^\varepsilon_1(s;t)^2\\ &\qquad+\big[q(s;t)\delta\sigma_x(s)+P(s;t)\sigma_x(s)\delta\sigma(s)+Q(s;t)\delta\sigma(s)\big]X^\varepsilon_1(s;t)\mathbb{1}_{E^\varepsilon_t}(s)\bigg\}ds\\ &\quad+\bigg\{p(s;t)\delta\sigma(s)\mathbb{1}_{E^\varepsilon_t}(s)\\ &\qquad+\big[p(s;t)\sigma_x(s)+q(s;t)\big]\big[X^\varepsilon_1(s;t)+X^\varepsilon_2(s;t)\big]\\ &\qquad+\tfrac12\big[p(s;t)\sigma_{xx}(s)+2P(s;t)\sigma_x(s)+Q(s;t)\big]X^\varepsilon_1(s;t)^2\\ &\qquad+\big[p(s;t)\delta\sigma_x(s)+P(s;t)\delta\sigma(s)\big]X^\varepsilon_1(s;t)\mathbb{1}_{E^\varepsilon_t}(s)\bigg\}dW(s) \end{aligned}$$

(see also Notation 2.1.1 and Remark 2.2.2). In particular, the maps g and G do not appear in the W-term.

For the sake of brevity, we denote, for any $s \in [t, T]$,

$$\begin{aligned}
A_1(s;t) &\coloneqq p(s;t)\delta b(s) + q(s;t)\delta\sigma(s) + \tfrac{1}{2}P(s;t)(\delta\sigma(s))^2, \\
A_2(s;t) &\coloneqq p(s;t)b_x(s) + q(s;t)\sigma_x(s) - g(s, p(s;t), q(s;t); t), \\
A_3(s;t) &\coloneqq p(s;t)b_{xx}(s) + q(s;t)\sigma_{xx}(s) + 2P(s;t)b_x(s) + 2Q(s;t)\sigma_x(s) \\
&\quad + P(s;t)\sigma_x(s)^2 - G(s, P(s;t), Q(s;t); t), \\
A_4(s;t) &\coloneqq q(s;t)\delta\sigma_x(s) + P(s;t)\sigma_x(s)\delta\sigma(s) + Q(s;t)\delta\sigma(s), \\
\Theta_1(s;t) &\coloneqq p(s;t)\sigma_x(s) + q(s;t), \\
\Theta_2(s;t) &\coloneqq p(s;t)\sigma_{xx}(s) + 2P(s;t)\sigma_x(s) + Q(s;t), \\
\Theta_3(s;t) &\coloneqq p(s;t)\delta\sigma_x(s) + P(s;t)\delta\sigma(s)
\end{aligned}$$

($p(s;t)\delta\sigma(s)$ is already short enough). We focus here on the coefficients of $\mathbb{1}_{E_t^\varepsilon}(\cdot)$.

Remark 2.2.3. $A_4(\cdot\,;t)X_1^\varepsilon(\cdot\,;t)\mathbb{1}_{E_t^\varepsilon}(\cdot) = o_{\varepsilon\downarrow 0}(\varepsilon)$, which does not hold true for $\Theta_3(\cdot\,;t)$ (see [22]).

Putting it all together, for any $s \in [t, T]$,

$$\begin{aligned}
Y^\varepsilon(s;t) &= h(X^\varepsilon(T);t) + \int_s^T f(r, X^\varepsilon(r), \boldsymbol{u}^\varepsilon(r), Y^\varepsilon(r;t), Z^\varepsilon(r;t);t)\,dr \\
&\quad - \int_s^T Z^\varepsilon(r;t)\,dW(r) \\
&= h(\bar{X}(T);t) + o_{\varepsilon\downarrow 0}(\varepsilon) + p(s;t)\big[X_1^\varepsilon(s;t) + X_2^\varepsilon(s;t)\big] + \tfrac{1}{2}P(s;t)X_1^\varepsilon(s;t)^2 \\
&\quad + \int_s^T \Big[f(r, X^\varepsilon(r), \boldsymbol{u}^\varepsilon(r), Y^\varepsilon(r;t), Z^\varepsilon(r;t);t) + A_1(r;t)\mathbb{1}_{E_t^\varepsilon}(r) \\
&\qquad + A_2(r;t)\big[X_1^\varepsilon(r;t) + X_2^\varepsilon(r;t)\big] + \tfrac{1}{2}A_3(r;t)X_1^\varepsilon(r;t)^2\Big]\,dr \\
&\quad - \int_s^T \Big[Z^\varepsilon(r;t) - \Big(p(r;t)\delta\sigma(r)\mathbb{1}_{E_t^\varepsilon}(r) + \Theta_1(r;t)\big[X_1^\varepsilon(r;t) + X_2^\varepsilon(r;t)\big] \\
&\qquad + \tfrac{1}{2}\Theta_2(r;t)X_1^\varepsilon(r;t)^2 + \Theta_3(r;t)X_1^\varepsilon(r;t)\mathbb{1}_{E_t^\varepsilon}(r)\Big)\Big]\,dW(r).
\end{aligned}$$

The key idea is to be able to narrow our analysis to the integral in dr. So, if we denote, for $\varepsilon \downarrow 0$,

$$\tilde{Y}^\varepsilon(\cdot\,;t) \coloneqq Y^\varepsilon(\cdot\,;t) - \Big(p(\cdot\,;t)\big[X_1^\varepsilon(\cdot\,;t) + X_2^\varepsilon(\cdot\,;t)\big] + \tfrac{1}{2}P(\cdot\,;t)X_1^\varepsilon(\cdot\,;t)^2\Big)$$

and

$$\tilde{Z}^{\varepsilon}(\cdot\,;t) \coloneqq Z^{\varepsilon}(\cdot\,;t) - \Big(p(\cdot\,;t)\delta\sigma(\cdot)\mathbb{1}_{E_t^{\varepsilon}}(\cdot) + \Theta_1(\cdot\,;t)\big[X_1^{\varepsilon}(\cdot\,;t) + X_2^{\varepsilon}(\cdot\,;t)\big] + \tfrac{1}{2}\Theta_2(\cdot\,;t)X_1^{\varepsilon}(\cdot\,;t)^2 + \Theta_3(\cdot\,;t)X_1^{\varepsilon}(\cdot\,;t)\mathbb{1}_{E_t^{\varepsilon}}(\cdot)\Big)$$

(on $[t,T]\times\Omega$), then we can write, for any $s\in[t,T]$,

$$\begin{aligned}\tilde{Y}^{\varepsilon}(s;t) &= h(\bar{X}(T);t) + o_{\varepsilon\downarrow 0}(\varepsilon)\\ &\quad + \int_s^T \Big[f(r, X^{\varepsilon}(r), \boldsymbol{u}^{\varepsilon}(r), Y^{\varepsilon}(r;t), Z^{\varepsilon}(r;t);t) + A_1(r;t)\mathbb{1}_{E_t^{\varepsilon}}(r)\\ &\qquad + A_2(r;t)\big[X_1^{\varepsilon}(r;t) + X_2^{\varepsilon}(r;t)\big] + \tfrac{1}{2}A_3(r;t)X_1^{\varepsilon}(r;t)^2\Big]\,dr\\ &\quad - \int_s^T \tilde{Z}^{\varepsilon}(r;t)\,dW(r).\end{aligned}$$

For $\varepsilon\downarrow 0$, we define an approximate variational pair process of first order of $(\bar{Y}(\cdot\,;t),\bar{Z}(\cdot\,;t))$ as

$$Y_1^{\varepsilon}(\cdot\,;t) \coloneqq \tilde{Y}^{\varepsilon}(\cdot\,;t) - \bar{Y}(\cdot\,;t)$$

and

$$Z_1^{\varepsilon}(\cdot\,;t) \coloneqq \tilde{Z}^{\varepsilon}(\cdot\,;t) - \bar{Z}(\cdot\,;t)$$

(on $[t,T]\times\Omega$), and we then get that, for any $s\in[t,T]$,

$$\begin{aligned}Y_1^{\varepsilon}(s;t) &= o_{\varepsilon\downarrow 0}(\varepsilon) + \int_s^T \Big[f(r, X^{\varepsilon}(r), \boldsymbol{u}^{\varepsilon}(r), Y^{\varepsilon}(r;t), Z^{\varepsilon}(r;t);t)\\ &\qquad - f(r, \bar{X}(r), \bar{\boldsymbol{u}}(r), \bar{Y}(r;t), \bar{Z}(r;t);t) + A_1(r;t)\mathbb{1}_{E_t^{\varepsilon}}(r)\\ &\qquad + A_2(r;t)\big[X_1^{\varepsilon}(r;t) + X_2^{\varepsilon}(r;t)\big] + \tfrac{1}{2}A_3(r;t)X_1^{\varepsilon}(r;t)^2\Big]\,dr\\ &\quad - \int_s^T Z_1^{\varepsilon}(r;t)\,dW(r).\end{aligned}$$

Remark 2.2.4. For $\Theta(\cdot\,;t)\in\mathcal{L}_{\mathbb{F}}^2(t,T;\mathbb{R})$, $\varepsilon\downarrow 0$ and $r\in[t,T]$,

$$\begin{aligned}&f(r, \bar{X}(r), \bar{\boldsymbol{u}}(r), \bar{Y}(r;t), \bar{Z}(r;t);t)\\ &\quad = f(r, \bar{X}(r), \bar{\boldsymbol{u}}^{\varepsilon}(r), \bar{Y}(r;t), \bar{Z}(r;t) + \Theta(r;t)\mathbb{1}_{E_t^{\varepsilon}}(r);t)\\ &\qquad + \big[f(r, \bar{X}(r), \bar{\boldsymbol{u}}(r), \bar{Y}(r;t), \bar{Z}(r;t);t)\\ &\qquad\quad - f(r, \bar{X}(r), \boldsymbol{u}(r), \bar{Y}(r;t), \bar{Z}(r;t) + \Theta(r;t);t)\big]\mathbb{1}_{E_t^{\varepsilon}}(r).\end{aligned}$$

Finally, for $\varepsilon \downarrow 0$, let $(\Delta Y^\varepsilon(\cdot\,;t), \Delta Z^\varepsilon(\cdot\,;t))$ be such that $\Delta Y^\varepsilon(T;t) = 0$, and, for any $s \in [t,T]$,

$$\begin{aligned}\Delta Y^\varepsilon(s;t) = \int_s^T \Big\{ & f_y(r;t)\Delta Y^\varepsilon(r;t) + f_z(r;t)\Delta Z^\varepsilon(r;t) \\ & + \Big[p(r;t)\delta b(r) + q(r;t)\delta\sigma(r) + \tfrac{1}{2}P(r;t)(\delta\sigma(r))^2 \\ & + f(r,\bar{X}(r),\boldsymbol{u}(r),\bar{Y}(r;t),\bar{Z}(r;t) + p(r;t)\delta\sigma(r);t) \\ & - f(r,\bar{X}(r),\bar{\boldsymbol{u}}(r),\bar{Y}(r;t),\bar{Z}(r;t);t)\Big]\mathbb{1}_{E_t^\varepsilon}(r)\Big\}\,dr \\ - \int_s^T & \Delta Z^\varepsilon(r;t)\,dW(r)\end{aligned}$$

(again, $\Delta Y^\varepsilon(t;t)$ is a deterministic constant). The following result is essentially Theorem 1 in [22].

Lemma 2.2.2 (Hu (2017)). *W.r.t. Notation 2.2.1,*

$$\begin{aligned} Y_1^\varepsilon(\cdot\,;t) - \Delta Y^\varepsilon(\cdot\,;t) &= o_{\varepsilon\downarrow 0}(\varepsilon), \\ Z_1^\varepsilon(\cdot\,;t) - \Delta Z^\varepsilon(\cdot\,;t) &= o_{\varepsilon\downarrow 0}(\varepsilon). \end{aligned}$$

As a remarkable corollary of Lemma 2.2.2, we get

$$Y^\varepsilon(t;t) - \bar{Y}(t;t) = \Delta Y^\varepsilon(t;t) + o_{\varepsilon\downarrow 0}(\varepsilon) \tag{2.15}$$

(as well as $Z^\varepsilon(t;t) = \bar{Z}(t;t) + \Delta Z^\varepsilon(t;t) + p(t;t)\delta\sigma(t) + o_{\varepsilon\downarrow 0}(\varepsilon)$). Furthermore, if $\kappa(\cdot\,;t)$ is the change of numéraire defined through (2.1) of Definition 2.1.1, then, for any $s \in [t,T]$,

$$d\langle \kappa(\cdot\,;t), \Delta Y^\varepsilon(\cdot\,;t)\rangle(s) = \kappa(s;t) f_z(s;t)\Delta Z^\varepsilon(s;t)ds,$$

and hence

$$\begin{aligned}\Delta Y^\varepsilon(t;t) = \mathbf{E}\Big[\int_t^T \kappa(r;t)\Big[& p(r;t)\delta b(r) + q(r;t)\delta\sigma(r) + \tfrac{1}{2}P(r;t)(\delta\sigma(r))^2 \\ & + f(r,\bar{X}(r),\boldsymbol{u}(r),\bar{Y}(r;t),\bar{Z}(r;t) + p(r;t)\delta\sigma(r);t) \\ & - f(r,\bar{X}(r),\bar{\boldsymbol{u}}(r),\bar{Y}(r;t),\bar{Z}(r;t);t)\Big]\mathbb{1}_{E_t^\varepsilon}(r)\,dr\Big]. \end{aligned} \tag{2.16}$$

In conclusion, for any $s \in [t,T]$, by (2.15) and (2.16), and by the classic Lebesgue differentiation theorem,

$$\begin{aligned}\liminf_{\varepsilon \downarrow 0} \frac{J(\bar{\boldsymbol{u}}^{\varepsilon}(\cdot);t,x) - J(\bar{\boldsymbol{u}}(\cdot);t,x)}{\varepsilon} &= \lim_{\varepsilon \downarrow 0} \frac{\Delta Y^{\varepsilon}(t;t)}{\varepsilon} \\ &= \mathbf{E}\big[\kappa(s;t)\delta\mathcal{H}(s;t,\boldsymbol{u}(s))\big]\end{aligned}$$

(see Definitions 2.1.4 and 2.1.5 and Notation 2.1.2) as it was our aim to prove.

Remark 2.2.5. $\kappa(t;t)$ could have been defined, in (2.1), as a constant > 0 not necessarily equal to 1.

Remark 2.2.6. We highlight the key difference with respect to the classical maximum principle, summarizing now in a few words what has just been seen technically (for completeness, see also Remark 1.2.14). The utility functional $J(\cdot\,;t,x)$ must be optimized in the ("weak") sense of equilibrium policies and, in particular, through the usual spike variation technique; so, on the one hand, a treatment similar to that initially formulated in [45, Chap. 3, Sect. 4] is set. On the other hand, $J(\cdot\,;t,x)$ has a precise ("strong") structure that derives from a recursive utility system which is a BSDE; therefore, to complete the calculations, the powerful techniques in [22] are taken up ad hoc.

2.3 Multidimensional case

Fix $d,k,m \in \mathbb{N}^*$ and let $W = (W(t))_{t\in[0,T]}$ be a d-dimensional Brownian motion, or Wiener process, on $(\Omega,\mathcal{F},\mathbf{P})$: for any $t \in [0,T]$,

$$W(t) = (W_1(t),\dots,W_d(t))^{\mathsf{T}} \in \mathbb{R}^d.$$

The state domain becomes a non-empty open pluri-interval

$$I \subseteq \mathbb{R}^k$$

and the independent variables change as follows: $x = (x_1,\dots,x_k)^{\mathsf{T}} \in \mathbb{R}^k$, $y = (y_1,\dots,y_m)^{\mathsf{T}} \in \mathbb{R}^m$ and

$$z = \left[z_1,\dots,z_m\right]^{\mathsf{T}} = \left[z^1,\dots,z^d\right] \in \mathbb{R}^{m\times d}$$

where, for $i = 1,\dots,m$,

$$z_i = (z_{i,1},\dots,z_{i,d}) \in \mathbb{R}^{1\times d}$$

and, for $j = 1,\dots,d$,

$$z^j = (z_{1,j},\dots,z_{m,j})^{\mathsf{T}} \in \mathbb{R}^m.$$

The coefficients, with the consequent regularity (see Assumption 1), change as follows:

$$b\colon [0,T]_s \times \mathbb{R}^k_x \times U_{\boldsymbol{u}} \to \mathbb{R}^k,$$

with $b(\boldsymbol{\cdot}) = (b_1(\boldsymbol{\cdot}), \ldots, b_k(\boldsymbol{\cdot}))^\mathsf{T} \in \mathbb{R}^k$;

$$\sigma\colon [0,T]_s \times \mathbb{R}^k_x \times U_{\boldsymbol{u}} \to \mathbb{R}^{k\times d}$$

with

$$\sigma(\boldsymbol{\cdot}) = [\sigma_1(\boldsymbol{\cdot}), \ldots, \sigma_k(\boldsymbol{\cdot})]^\mathsf{T} = \left[\sigma^1(\boldsymbol{\cdot}), \ldots, \sigma^d(\boldsymbol{\cdot})\right] \in \mathbb{R}^{k\times d}$$

where, for $i = 1, \ldots, k$,

$$\sigma_i(\boldsymbol{\cdot}) = (\sigma_{i,1}(\boldsymbol{\cdot}), \ldots, \sigma_{i,d}(\boldsymbol{\cdot})) \in \mathbb{R}^{1\times d}$$

and, for $j = 1, \ldots, d$,

$$\sigma^j(\boldsymbol{\cdot}) = (\sigma_{1,j}(\boldsymbol{\cdot}), \ldots, \sigma_{k,j}(\boldsymbol{\cdot}))^\mathsf{T} \in \mathbb{R}^k;$$

$$f\colon [0,T]_s \times \mathbb{R}^k_x \times U_{\boldsymbol{u}} \times \mathbb{R}^m_y \times \mathbb{R}^{m\times d}_z \times [0,T[_t \to \mathbb{R}^m,$$

with $f(\boldsymbol{\cdot}\,;t) = (f_1(\boldsymbol{\cdot}\,;t), \ldots, f_m(\boldsymbol{\cdot}\,;t))^\mathsf{T} \in \mathbb{R}^m$, and

$$h\colon \mathbb{R}^k_x \times [0,T[_t \to \mathbb{R}^m$$

with $h(\boldsymbol{\cdot}\,;t) = (h_1(\boldsymbol{\cdot}\,;t), \ldots, h_m(\boldsymbol{\cdot}\,;t))^\mathsf{T} \in \mathbb{R}^m$.

The state equation (1.3) becomes

$$\begin{cases} dX(s) = b(s, X(s), \boldsymbol{u}(s))ds + \sum_{j=1}^d \sigma^j(s, X(s), \boldsymbol{u}(s))dW_j(s), \\ X(t) = x, \end{cases}$$

(where $s \in [t,T]$) and the recursive (dis)utility system (1.5) becomes

$$\begin{cases} dY(s;t) = -f(s, X(s), \boldsymbol{u}(s), Y(s;t), Z(s;t);t)ds + \sum_{j=1}^d Z^j(s;t)dW_j(s), \\ Y(T;t) = h(X(T);t), \end{cases}$$

(where $s \in [t,T]$). Regarding the (dis)utility or cost functional as in (1.8), take

$$\gamma\colon \mathbb{R}^m_y \times [0,T[_t \to \mathbb{R}$$

of (differentiability) class C^1 w.r.t. the variable $y \in \mathbb{R}^m$ and set

$$J(\boldsymbol{u}(\boldsymbol{\cdot}); t, x) \doteq \gamma(Y(t;t);t).$$

Now, for any ($t \in [0,T[$, $x \in I$ and) admissible 4-tuple

$$(\bar{\boldsymbol{u}}(\boldsymbol{\cdot}), \bar{X}(\boldsymbol{\cdot}), \bar{Y}(\boldsymbol{\cdot}\,;t), \bar{Z}(\boldsymbol{\cdot}\,;t))$$

the equation (2.2) becomes the multidimensional adjoint equation of first order associated with $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ determined, for $i = 1, \dots, m$, by

$$\begin{cases} dp^i(s;t) = -g_i(s, p(s;t), q(s;t); t)ds + \sum_{j=1}^{d} q_i^j(s;t) dW_j(s), \\ p^i(T;t) = \mathrm{D}_x h_i(\bar{X}(T); t), \end{cases}$$

(where $s \in [t, T]$ and) where:

$$p(\cdot\,;t) = \left[p^1(\cdot\,;t), \dots, p^m(\cdot\,;t)\right] \in \mathbb{R}^{k\times m}$$

with, for any $i = 1, \dots, m$, $p^i(\cdot\,;t) = (p_{1,i}(\cdot\,;t), \dots, p_{k,i}(\cdot\,;t))^\intercal \in \mathbb{R}^k$;

$$q(\cdot\,;t) = \left[q^1(\cdot\,;t), \dots, q^d(\cdot\,;t)\right] \in \mathbb{R}^{(k\times m)\times d}$$

where, for $j = 1, \dots, d$,

$$q^j(\cdot\,;t) = \left[q_1^j(\cdot\,;t), \dots, q_m^j(\cdot\,;t)\right] \in \mathbb{R}^{k\times m}$$

and, for $i = 1, \dots, m$,

$$q_i^j(\cdot\,;t) = (q_{1,i}^j(\cdot\,;t), \dots, q_{k,i}^j(\cdot\,;t))^\intercal \in \mathbb{R}^k;$$

for $i = 1, \dots, m$, the map

$$g_i \colon [0, T]_s \times \mathbb{R}_p^{k\times m} \times \mathbb{R}_q^{(k\times m)\times d} \times [0, T[_t \to \mathbb{R}^k$$

is given as in [22]. Similarly, the equation (2.4) becomes the multidimensional adjoint equation of second order associated with $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ determined, for $i = 1, \dots, m$, by

$$\begin{cases} dP^i(s;t) = -G_i(s, P(s;t), Q(s;t); t)ds + \sum_{j=1}^{d} Q_i^j(s;t) dW_j(s), \\ P^i(T;t) = \mathrm{D}_{xx} h_i(\bar{X}(T); t), \end{cases}$$

(where $s \in [t, T]$ and) where:

$$P(\cdot\,;t) = \left[P^1(\cdot\,;t), \dots, P^m(\cdot\,;t)\right] \in \mathbb{R}^{k^2\times m}$$

with, for any $i = 1, \dots, m$, $P^i(\cdot\,;t) = [P_{1,i}(\cdot\,;t), \dots, P_{k,i}(\cdot\,;t)]^\intercal \in \mathbb{R}^{k^2}$;

$$Q(\cdot\,;t) = \left[Q^1(\cdot\,;t), \dots, Q^d(\cdot\,;t)\right] \in \mathbb{R}^{(k^2\times m)\times d}$$

where, for $j = 1, \dots, d$,

$$Q^j(\cdot\,;t) = \left[Q_1^j(\cdot\,;t), \dots, Q_m^j(\cdot\,;t)\right] \in \mathbb{R}^{k^2\times m}$$

and, for $i = 1, \ldots, m$,

$$Q_i^j(\cdot\,;t) = \big[Q_{1,i}^j(\cdot\,;t), \ldots, Q_{k,i}^j(\cdot\,;t)\big]^\top \in \mathbb{R}^{k^2};$$

for $i = 1, \ldots, m$, the map

$$G_i\colon [0,T]_s \times \mathbb{R}_P^{k^2\times m} \times \mathbb{R}_Q^{(k^2\times m)\times d} \times [0,T[_t \to \mathbb{R}^{k^2}$$

is given as in [22].

Next, the process $\kappa(\cdot\,;t) \in \mathbb{R}^m$ associated with $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ becomes the solution of

$$\begin{cases} d\kappa(s;t) = \big(\mathrm{D}_y f(s;t)\big)^\top \kappa(s;t)ds + \sum_{j=1}^d \big(\mathrm{D}_{z^j} f(s;t)\big)^\top \kappa(s;t)dW_j(s), \\ \kappa(t;t) = \mathrm{D}_y \gamma(\bar{Y}(t;t);t), \end{cases}$$

(where $s \in [t,T]$) while the generalized Hamiltonian (function) of first order (2.6) associated with b, σ, f becomes

$$H\colon [0,T]_s \times \mathbb{R}_x^k \times U_{\boldsymbol{u}} \times \mathbb{R}_y^m \times \mathbb{R}_z^{m\times d} \times \mathbb{R}_p^{k\times m} \times \mathbb{R}_q^{(k\times m)\times d} \times [0,T[_t \times I_{\bar{x}} \times U_{\bar{\boldsymbol{u}}} \to \mathbb{R}^m$$

given by

$$\begin{aligned} H(s,x,\boldsymbol{u},y,z,p,q;t,\bar{x},\bar{\boldsymbol{u}}) &\doteq p^\top b(s,x,\boldsymbol{u}) + \textstyle\sum_{j=1}^d (q^j)^\top \sigma^j(s,x,\boldsymbol{u}) \\ &\qquad + f\big(s,x,\boldsymbol{u},y,z + p^\top[\sigma(s,x,\boldsymbol{u}) - \sigma(s,\bar{x},\bar{\boldsymbol{u}})];t\big) \end{aligned}$$

where

$$q = \big[q^1, \ldots, q^d\big] \in \mathbb{R}^{(k\times m)\times d}$$

where, for $j = 1, \ldots, d$,

$$q^j = \big[q_1^j, \ldots, q_m^j\big] \in \mathbb{R}^{k\times m}$$

and, for $i = 1, \ldots, m$,

$$q_i^j = (q_{1,i}^j, \ldots, q_{k,i}^j)^\top \in \mathbb{R}^k$$

and the generalized Hamiltonian (function) of second order (2.7) associated with b, σ, f becomes

$$\mathcal{H}\colon$$
$$[0,T]_s \times \mathbb{R}_x^k \times U_{\boldsymbol{u}} \times \mathbb{R}_y^m \times \mathbb{R}_z^{m\times d} \times \mathbb{R}_p^{k\times m} \times \mathbb{R}_q^{(k\times m)\times d} \times \mathbb{R}_P^{k^2\times m} \times [0,T[_t \times I_{\bar{x}} \times U_{\bar{\boldsymbol{u}}} \to \mathbb{R}^m$$

given by

$$\begin{aligned} \mathcal{H}(s,x,\boldsymbol{u},y,z,p,q,P;t,\bar{x},\bar{\boldsymbol{u}}) &\doteq H(s,x,\boldsymbol{u},y,z,p,q;t,\bar{x},\bar{\boldsymbol{u}}) \\ &+ \tfrac{1}{2} \textstyle\sum_{j=1}^d [\sigma^j(s,x,\boldsymbol{u}) - \sigma^j(s,\bar{x},\bar{\boldsymbol{u}})]^\top P^\top [\sigma^j(s,x,\boldsymbol{u}) - \sigma^j(s,\bar{x},\bar{\boldsymbol{u}})] \end{aligned}$$

according to the following notation: for $j = 1, \dots, d$,

$$[\sigma^j(s, x, \boldsymbol{u}) - \sigma^j(s, \bar{x}, \bar{\boldsymbol{u}})]^\mathsf{T} P^\mathsf{T} [\sigma^j(s, x, \boldsymbol{u}) - \sigma^j(s, \bar{x}, \bar{\boldsymbol{u}})] \in \mathbb{R}^m$$

having, for $i = 1, \dots, m$, i-th component given by

$$[\sigma^j(s, x, \boldsymbol{u}) - \sigma^j(s, \bar{x}, \bar{\boldsymbol{u}})]^\mathsf{T} (P^i)^\mathsf{T} [\sigma^j(s, x, \boldsymbol{u}) - \sigma^j(s, \bar{x}, \bar{\boldsymbol{u}})]$$

where

$$P = \left[P^1, \dots, P^m\right] \in \mathbb{R}^{k^2 \times m}$$

with, for any $i = 1, \dots, m$, $P^i = [P_{1,i}, \dots, P_{k,i}]^\mathsf{T} \in \mathbb{R}^{k^2}$. Regarding Notation 2.1.2,

$$\begin{aligned}
&\mathcal{H}(s;t) \\
&\quad := \mathcal{H}(s, \bar{X}(s), \bar{\boldsymbol{u}}(s), \bar{Y}(s;t), \bar{Z}(s;t), p(s;t), q(s;t), P(s;t); t, \bar{X}(s), \bar{\boldsymbol{u}}(s))
\end{aligned}$$

and, for $\boldsymbol{u} \in U$,

$$\begin{aligned}
&\delta\mathcal{H}(s;t,\boldsymbol{u}) \\
&:= \mathcal{H}(s, \bar{X}(s), \boldsymbol{u}, \bar{Y}(s;t), \bar{Z}(s;t), p(s;t), q(s;t), P(s;t); t, \bar{X}(s), \bar{\boldsymbol{u}}(s)) - \mathcal{H}(s;t).
\end{aligned}$$

Finally, for $\varepsilon \downarrow 0$, let $(\Delta Y^\varepsilon(\,\cdot\,;t), \Delta Z^\varepsilon(\,\cdot\,;t))$ be with values in $\mathbb{R}^m \times \mathbb{R}^{m \times d}$ and $\Delta Y^\varepsilon(T;t) = 0$ such that, for any $s \in [t, T]$,

$$\begin{aligned}
\Delta Y^\varepsilon(s;t) = \int_s^T \Big\{ &\mathrm{D}_y f(r;t) \Delta Y^\varepsilon(r;t) + \textstyle\sum_{j=1}^d \mathrm{D}_{z^j} f(r;t) \Delta Z_j^\varepsilon(r;t) \\
&+ \Big[p(r;t)^\mathsf{T} \delta b(r) + \textstyle\sum_{j=1}^d q^j(r;t)^\mathsf{T} \delta\sigma^j(r) \\
&+ \tfrac{1}{2} \textstyle\sum_{j=1}^d (\delta\sigma^j(r))^\mathsf{T} P(r;t)^\mathsf{T} \delta\sigma^j(r) \\
&+ f\big(r, \bar{X}(r), \boldsymbol{u}(r), \bar{Y}(r;t), \bar{Z}(r;t) + p(r;t)^\mathsf{T} \delta\sigma(r); t\big) \\
&- f(r, \bar{X}(r), \bar{\boldsymbol{u}}(r), \bar{Y}(r;t), \bar{Z}(r;t); t) \Big] \mathbb{1}_{E_t^\varepsilon}(r) \Big\} dr \\
- \sum_{j=1}^d \int_s^T \Delta Z_j^\varepsilon(r;t)\, dW_j(r)
\end{aligned}$$

(for $j = 1, \dots, d$, $\Delta Z_j^\varepsilon(\,\cdot\,;t) \in \mathbb{R}^m$ is the j-th column of $\Delta Z^\varepsilon(\,\cdot\,;t)$). Then, as in the one-dimensional case, it could be shown that

$$J(\bar{\boldsymbol{u}}^\varepsilon(\cdot); t, x) - J(\bar{\boldsymbol{u}}(\cdot); t, x) = \big(\mathrm{D}_y \gamma(\bar{Y}(t;t); t)\big)^\mathsf{T} \Delta Y^\varepsilon(t;t) + o_{\varepsilon\downarrow 0}(\varepsilon)$$

(in $\mathbb{R}$) and that, here,

$$\begin{aligned}\big(\mathrm{D}_y\gamma(\bar Y(t;t);t)\big)^{\mathsf T}\Delta Y^\varepsilon(t;t) = \mathbf{E}\bigg[\int_t^T \kappa(r;t)^{\mathsf T}\Big[p(r;t)^{\mathsf T}\delta b(r)\\ + \textstyle\sum_{j=1}^d q^j(r;t)^{\mathsf T}\delta\sigma^j(r) + \frac{1}{2}\sum_{j=1}^d(\delta\sigma^j(r))^{\mathsf T}P(r;t)^{\mathsf T}\delta\sigma^j(r)\\ + f\big(r,\bar X(r),\boldsymbol{u}(r),\bar Y(r;t),\bar Z(r;t)+p(r;t)^{\mathsf T}\delta\sigma(r);t\big)\\ - f(r,\bar X(r),\bar{\boldsymbol{u}}(r),\bar Y(r;t),\bar Z(r;t);t)\Big]\mathbb{1}_{E_t^\varepsilon}(r)\,dr\bigg].\end{aligned}$$

Therefore, Lemma 2.1.1 changes as follows.

Lemma 2.3.1. *Fix $\boldsymbol{u}(\cdot)\in\mathscr{U}[t,T]$. For $\varepsilon\in\,]0,T-t[$ and $s\in[t,T]$, let $E_t^\varepsilon\equiv E_{t,s}^\varepsilon$ be given by*

$$E_t^\varepsilon := \begin{cases}[s,s+\varepsilon], & \text{if } s<T,\\ [T-\varepsilon,T], & \text{if } s=T,\end{cases}$$

and let $\bar{\boldsymbol{u}}^\varepsilon(\cdot)$ be the spike variation of $\bar{\boldsymbol{u}}(\cdot)$ w.r.t. $\boldsymbol{u}(\cdot)$ and E_t^ε. Then, for any $s\in[t,T]$, the $\liminf_{\varepsilon\downarrow 0}$ as in (1.11) is an actual limit and takes the form

$$\liminf_{\varepsilon\downarrow 0}\frac{J(\bar{\boldsymbol{u}}^\varepsilon(\cdot);t,x)-J(\bar{\boldsymbol{u}}(\cdot);t,x)}{\varepsilon} = \mathbf{E}\big[\kappa(s;t)^{\mathsf T}\delta\mathcal{H}(s;t,\boldsymbol{u}(s))\big].$$

Consequently, Theorem 2.1.1 changes as follows (and the proof is essentially the same).

Theorem 2.3.1 (Maximum principle)**.** *Suppose there exists a measurable map $\boldsymbol{\Pi}\colon[0,T]\times I\to U$ such that, for any $s\in[t,T]$ (and $\mathbf{P}$-a.s.),*

$$\bar{\boldsymbol{u}}(s)=\boldsymbol{\Pi}(s,\bar X(s)).$$

Then the following three conditions are equivalent.

1. *$\boldsymbol{\Pi}$ is an equilibrium policy, i.e., $(\bar{\boldsymbol{u}}(\cdot),\bar X(\cdot),\bar Y(\cdot\,;t),\bar Z(\cdot\,;t))$ is an equilibrium 4-tuple.*
2. *For any $\boldsymbol{u}\in U$ (and $\mathbf{P}$-a.s.),*
$$\big(\mathrm{D}_y\gamma(\bar Y(t;t);t)\big)^{\mathsf T}\delta\mathcal{H}(t;t,\boldsymbol{u})\geq 0.$$
3. *If $\bar Z(t;t)$ and $q(t;t)$ are deterministic constants, then*
$$\begin{aligned}\boldsymbol{\Pi}(t,x)\in\operatorname*{arg\,min}_{\boldsymbol{u}\in U}\big(\mathrm{D}_y\gamma(\bar Y(t;t);t)\big)^{\mathsf T}\cdot\\ \cdot\,\mathcal{H}(t,x,\boldsymbol{u},\bar Y(t;t),\bar Z(t;t),p(t;t),q(t;t),P(t;t);t,x,\boldsymbol{\Pi}(t,x)).\end{aligned}$$

2.4 Application to portfolio management (I)

Starting from the financial market model, set $r(\cdot) \equiv r \in\,]0,\infty[$ as a constant risk-free interest rate on $[0,T]$ and let $S_0 = (S_0(s))_{s\in[0,T]}$ be the value of a (deterministic) savings account, or *bond*, which accrues interest at the rate r (with $S_0(0) \in\,]0,\infty[$ exogenously specified): i.e., for $s \in [0,T]$,

$$dS_0(s) = r\,S_0(s)\,ds,$$

i.e., $S_0(s) = S_0(0)\,e^{rs}$. Let $S = (S(s))_{s\in[0,T]}$ be the price of a risky asset, or *stock*, that follows a geometric Brownian motion in dimension $d = 1$ (with $S(0) \in\,]0,\infty[$ exogenously specified): i.e., for $s \in [0,T]$,

$$dS(s) = S(s)\big[\varrho\,ds + \sigma\,dW(s)\big],$$

i.e., explicitly,

$$S(s) = S(0)\exp\Big\{(\varrho - \sigma^2/2)\,s + \sigma\,W(s)\Big\},$$

where $\varrho \in\,]r,\infty[$ is the mean rate of return by the stock and $\sigma \in\,]0,\infty[$ is the volatility of the price (both of which are constant scalars). We denote the excess return, on average, by investment in the stock as

$$\mu := \varrho - r\,.$$

For this, as well as for the following discussion, see, e.g., [14] and [13].

Remark 2.4.1. The above market is assumed to be complete (there are no sources of randomness other than the stock).

A decision-maker within this market, or agent for short, is assumed to invest her/his wealth $X(\cdot)$ in the stock and the bond and to consume continuously over time (on $[0,T]$). So, for $s \in [0,T]$, let $\zeta(s)$ be the proportion of current wealth $X(s)$ invested in the stock at time s (with a sign) and let $c(s)$ be the proportion of $X(s)$ consumed at time s. Then $n = 2$, the control domain becomes

$$U = [-1,1]_\zeta \times [0,1]_c,$$

and any admissible control $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$ is a portfolio strategy that can be written as

$$\boldsymbol{u}(\cdot) = (\zeta(\cdot), c(\cdot))$$

(see Definition 1.2.1) and which we refer to as the *investment-consumption policy*.

Remark 2.4.2. For $s \in [0,T]$, $1 - \zeta(s)$ coincides with the proportion of $X(s)$ invested in the bond.

We accept as true the usual self-financing condition, namely, that the variation in wealth over time is due exclusively to profits and losses from investing in the stock and from consumption (there is no cashflow coming in or out), and we consider investment-consumption policies in feedback form:

$$(\zeta(\cdot), c(\cdot)) \equiv (\zeta(\cdot\,, X(\cdot)), c(\cdot\,, X(\cdot))). \tag{2.17}$$

We assume that the agent derives utility, from intertemporal consumption $c(\cdot)X(\cdot)$ and final wealth $X(T)$, which she/he tries to optimize by minimizing in a sense a discounted expectation involving (dis)utility functions. Therefore, let $\upsilon(\cdot)$ and $\hat{\upsilon}(\cdot)$ be two scalar functions of a real variable that satisfy the classical Uzawa–Inada conditions (utility functions, in fact): i.e.,

$$\upsilon\colon [0,\infty[\, \to [0,\infty[$$

is of class C^2 and strictly increasing such that, for any $x \in\,]0,\infty[$, $\upsilon''(x) < 0$ (thus, $\upsilon(\cdot)$ is strictly convex on $]0,\infty[$), with $\upsilon(0) = 0$ and $\lim_{x\downarrow 0} \upsilon'(x) = \infty$, $\lim_{x\uparrow\infty} \upsilon'(x) = 0$ (and the same for $\hat{\upsilon}(\cdot)$).

Remark 2.4.3. The marginal disutility function $\upsilon'\colon [0,\infty[\, \to\,]0,\infty[$ is of class C^1 and bijective outside the origin, and has an inverse function

$$(\upsilon')^{-1}\colon\,]0,\infty[\, \to\,]0,\infty[$$

that is continuous and strictly decreasing with again $\lim_{x\downarrow 0}(\upsilon')^{-1}(x) = \infty$, $\lim_{x\uparrow\infty}(\upsilon')^{-1}(x) = 0$; in particular, $(-\upsilon')^{-1}(\cdot)$ is a positive function with domain $]-\infty, 0[$ (and the same for $\hat{\upsilon}'(\cdot)$).

Notation 2.4.1. We write

$$\Upsilon \coloneqq (-\upsilon')^{-1}.$$

Furthermore, for fixed $t \in [0,T[$, let $\hbar(\cdot\,;t)$ and $\hat{\hbar}(\cdot\,;t)$ be two discount functions on $[t,T]$: i.e.,

$$\hbar(\cdot\,;t)\colon [t,T] \to\,]0,\infty[$$

with $\hbar(t;t) = 1$ and $\int_t^T \hbar(s;t)\,ds < \infty$ (and the same for $\hat{\hbar}(\cdot\,;t)$).

Remark 2.4.4. If $\hbar(\cdot\,;t)$ is differentiable, then the corresponding (psychological) discount rate

$$-\hbar'(\cdot\,;t)/\hbar(\cdot\,;t)$$

i.e., the rate of return used to discount future cashflows back to their present value, can be considered to be a monotonic function (and the same for $\hat{\hbar}(\cdot\,;t)$).

Example 2.4.1. Regarding $\upsilon(\cdot)$, we could take $\lambda \in]0,1[$ and, for $x \in [0,\infty[$,

$$\upsilon(x) \equiv \upsilon_\lambda(x) := x^\lambda/\lambda$$

thus obtaining a constant relative risk aversion equal to $1-\lambda \equiv -x\upsilon''(x)/\upsilon'(x)$ (and similarly for $\hat{\upsilon}(\cdot)$). Regarding $\hbar(\cdot\,;t)$, we could take a common discount function $\hbar(\cdot)$ on $[0,T-t[$ and, for $s \in [t,T]$,

$$\hbar(s;t) := \hbar(s-t),$$

where, for instance, associated with $K \in]0,\infty[$, we could imagine that, for $\tau \in [0,T-t[$,

$$\hbar(\tau) \equiv \hbar_K(\tau) := \frac{1}{1+K\tau}.$$

and then have to deal with hyperbolic discounting and non-constant discount rate (and similarly for $\hat{\hbar}(\cdot\,;t)$).

Remark 2.4.5. A non-exponential discount mechanism such as the one in Example 2.4.1 is hopelessly time-inconsistent, meaning that, given a cashflow $C(\cdot)$ on $]0,T]$ whose utility through $\upsilon(\cdot)$ must be actualized, the following hypothetical graph becomes inconsistent for any triplet $0 \le t < s < \tau \le T$:

$$\begin{array}{ccccc}
 & & \xleftarrow{\quad\quad\quad\tau-t\quad\quad\quad} & & \\
t & \xleftarrow{s-t} & s & \xleftarrow{\tau-s} & \tau \\
 & & & & \\
\hbar(s-t)\upsilon(C(s)) & \xleftarrow[\hbar(\cdot)]{} & \upsilon(C(s)) & & \\
 & & \hbar(\tau-s)\upsilon(C(\tau)) & \xleftarrow[\hbar(\cdot)]{} & \upsilon(C(\tau)) \\
\hbar(\tau-t)\upsilon(C(\tau)) & & \xleftarrow[\quad\quad\quad\hbar(\cdot)\quad\quad\quad]{} & & \upsilon(C(\tau))
\end{array}$$

Choosing $k=m=1$, as the state domain

$$I =]0,\infty[,$$

a fixed $t \in [0,T[$, positive processes

$$\beta(\cdot\,;t), \gamma(\cdot\,;t) \in \mathcal{L}^\infty_{\mathbb{F}}(t,T;\mathbb{R}),$$

$x \in I$ and $(\zeta(\cdot), c(\cdot)) \in \mathscr{U}[t,T]$, we consider the recursive stochastic control problem to be as follows:

$$\begin{cases} dX(s) = X(s)\big[\big(r + \mu\zeta(s) - c(s)\big)ds + \sigma\zeta(s)dW(s)\big], \\ dY(s;t) = -\hbar(s;t)\big[-\upsilon(c(s)X(s)) - \beta(s;t)Y(s;t) - \gamma(s;t)Z(s;t)\big]\,ds \\ \qquad\qquad + Z(s;t)dW(s), \\ X(t) = x, \quad Y(T;t) = -\hat{\hbar}(T;t)\hat{\upsilon}(X(T)), \end{cases}$$

(where $s \in [t,T]$), bearing in mind (2.17) (see Definition 1.2.5). See, e.g., [17].

Remark 2.4.6. By virtue of Remark 1.3.4, also fundamental in the short term,

$$Y(\cdot\,;t) < 0$$

($\mathbf{P}$-a.s.).

It is quite simple to verify that the sufficient conditions of Theorem 2.1.1, or rather of Corollary 2.1.2, can be expressed as follows.

Theorem 2.4.1. *Suppose there exists a measurable map*

$$\mathbf{\Pi} = (\Pi_1, \Pi_2)\colon [0,T]_s \times \,]0,\infty[_x \to [-1,1] \times [0,1]$$

of class C^2 w.r.t. the variable x with bounded first and second derivatives, even if multiplied by the identity function, and assume that if, for any $t \in [0,T[$ and $x \in I$, $\bar{X}(\cdot)$ is the solution of the FSDE

$$\begin{cases} dX(s) = X(s)\Big[\big(r + \mu\Pi_1(s,X(s)) - \Pi_2(s,X(s))\big)ds + \sigma\Pi_1(s,X(s))dW(s)\Big], \\ X(t) = x, \end{cases}$$

(where $s \in [t,T]$) then there exists a pair solution $(p(\cdot\,;t), q(\cdot\,;t))$ of the BSDE

$$\begin{cases} dp(s;t) = \Big\{\Big\{r + \mu\big[\Pi_1(s,\bar{X}(s)) + \bar{X}(s)\frac{\partial}{\partial x}\Pi_1(s,\bar{X}(s))\big] - \big[\Pi_2(s,\bar{X}(s)) \\ \qquad + \bar{X}(s)\frac{\partial}{\partial x}\Pi_2(s,\bar{X}(s))\big] - \sigma\hbar(s;t)\gamma(s;t)\big[\Pi_1(s,\bar{X}(s)) \\ \qquad + \bar{X}(s)\frac{\partial}{\partial x}\Pi_1(s,\bar{X}(s))\big] - \hbar(s;t)\beta(s;t)\Big\}p(s;t) \\ \quad + \Big\{\sigma\big[\Pi_1(s,\bar{X}(s)) + \bar{X}(s)\frac{\partial}{\partial x}\Pi_1(s,\bar{X}(s))\big] - \hbar(s;t)\gamma(s;t)\Big\}q(s;t) \\ \quad - \hbar(s;t)\upsilon'(\Pi_2(s,\bar{X}(s))\bar{X}(s))\big[\Pi_2(s,\bar{X}(s)) \\ \qquad + \bar{X}(s)\frac{\partial}{\partial x}\Pi_2(s,\bar{X}(s))\big]\Big\}ds \\ \quad + q(s;t)dW(s), \\ p(T;t) = -\hat{\hbar}(T;t)\upsilon'(\bar{X}(T)), \end{cases}$$

(where $s \in [t, T]$) such that, for any $s \in [t, T]$,

$$\begin{cases} \big[\mu - \sigma\gamma(s;t)\big]p(s;t) + \sigma q(s;t) = 0, \\ p(s;t) = -\hbar(s;t)\upsilon'(\Pi_2(s, \bar{X}(s))\bar{X}(s)). \end{cases}$$

Then $\mathbf{\Pi}$ is an equilibrium policy having, in particular,

$$\Pi_2(t, x) = \frac{\Upsilon\big(p(t;t)\big)}{x}.$$

Remark 2.4.7. Generalization to the case of multiple stocks would basically be a matter of formalization.

Chapter 3

Constrained problem: necessary conditions

In this chapter, we solve Problem 2 by adapting the computations of [45, Chap. 3, Sect. 6]. We suppose that Assumptions 1 and 2 hold, and we fix $t \in [0, T[$ and $x \in I$.

3.1 A maximum principle

We present the second of our main results, referring particularly to Definition 1.2.9 and annexes (see also Definitions 2.1.2, 2.1.3, 2.1.4, and 2.1.5 and Notation 2.1.2).

Theorem 3.1.1 (Maximum principle)**.** *Suppose that U is bounded. Let*

$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,; t), \bar{Z}(\cdot\,; t))$$

be an equilibrium 4-tuple such that $(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot))$ satisfies the state constraint, i.e.,

$$\boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot); t, x) \equiv \bar{\boldsymbol{Y}}(t; t) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}.$$

Then there exist two multipliers $\psi, \boldsymbol{\psi} \in [-1, 1]$ *with*

$$\psi^2 + \boldsymbol{\psi}^2 = 1 \tag{3.1}$$

such that the following two conditions hold.

1. *(Transversality condition) For any $\bar{v} \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$,*

$$\boldsymbol{\psi}\big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot); t, x)\big] \leq 0. \tag{3.2}$$

2. *Let* $(p(\cdot\,;t), q(\cdot\,;t))$, $(P(\cdot\,;t), Q(\cdot\,;t))$, $\mathcal{H}(\cdot\,;t)$, $\delta\mathcal{H}(\cdot\,;t, \text{-})$ *correspond to the 4-tuple*
$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$$
and $(\boldsymbol{p}(\cdot\,;t), \boldsymbol{q}(\cdot\,;t))$, $(\boldsymbol{P}(\cdot\,;t), \boldsymbol{Q}(\cdot\,;t))$, $\boldsymbol{\mathcal{H}}(\cdot\,;t)$, $\delta\boldsymbol{\mathcal{H}}(\cdot\,;t, \text{-})$ *correspond to the 4-tuple*
$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{\boldsymbol{Y}}(\cdot\,;t), \bar{\boldsymbol{Z}}(\cdot\,;t)).$$
Then, for any $\boldsymbol{u} \in U$ *(and* $\mathbf{P}$*-a.s.),*
$$\psi\,\delta\mathcal{H}(t;t,\boldsymbol{u}) + \boldsymbol{\psi}\,\delta\boldsymbol{\mathcal{H}}(t;t,\boldsymbol{u}) \geq 0. \tag{3.3}$$

Remark 3.1.1. Regarding Theorem 3.1.1, we point out the following.

- The condition (3.1) leads to the fact that ψ and $\boldsymbol{\psi}$ cannot both be zero. When $\psi \neq 0$ (i.e., $|\boldsymbol{\psi}| < 1$), we could say that the *qualification condition* is satisfied.
- The condition (3.2) means that, for any $\bar{v} \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$, $\boldsymbol{\psi}$ and $\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot);t,x)$ have opposite signs. Therefore, if $\boldsymbol{\psi} \neq 0$ (i.e., $|\psi| < 1$), the following two cases occur:
$$\boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot);t,x) = \begin{cases} \min \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}, & \text{if } \boldsymbol{\psi} < 0, \\ \max \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}, & \text{if } \boldsymbol{\psi} > 0. \end{cases}$$
- If $|\psi| = 1$ (i.e., $\boldsymbol{\psi} = 0$), then the condition (3.3) coincides with the necessary condition part of Theorem 2.1.1. To be precise, if $\psi = -1$, then, for any $\boldsymbol{u} \in U$ (and $\mathbf{P}$-a.s.),
$$\delta\mathcal{H}(t;t,\boldsymbol{u}) = 0,$$
i.e., the $\liminf_{\varepsilon\downarrow 0}$ in (1.11) is zero (see also Lemma 2.1.1).

3.2 A proof of Theorem 3.1.1

We recall below two preliminary results that we use: the first, Lemma 3.2.1, is Corollary 6.3 in [45] and is essentially *Ekeland's variational principle* (see also [9]); the second, Lemma 3.2.2, is the one-dimensional version of Lemma 6.5 in [45] and deals with the *distance function* to a subset of $\mathbb{R}$.

Lemma 3.2.1. *Let* $(V, \mathbf{d})$ *be a complete metric space and* $F\colon V \to \mathbb{R}$ *be a lower-semicontinuous map, w.r.t. the metric* $\mathbf{d}$*, which is bounded from below. Then, for each of those* $\varrho \in \,]0,\infty[$ *and* $\bar{\boldsymbol{v}} \in V$ *with*
$$F(\bar{\boldsymbol{v}}) \leq \inf_{\boldsymbol{v}\in V} F(\boldsymbol{v}) + \varrho,$$
there exists $\bar{\boldsymbol{v}}_\varrho \in V$ *such that the following three conditions hold.*

1. $F(\bar{\boldsymbol{v}}_\varrho) \leq F(\bar{\boldsymbol{v}})$.

2. $\mathbf{d}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{v}}_\varrho) \leq \sqrt{\varrho}$.

3. *For any* $\boldsymbol{v} \in V$,
$$F(\boldsymbol{v}) - F(\bar{\boldsymbol{v}}_\varrho) \geq -\sqrt{\varrho}\,\mathbf{d}(\boldsymbol{v}, \bar{\boldsymbol{v}}_\varrho). \tag{3.4}$$

Lemma 3.2.2. *Let* $\Gamma \subseteq \mathbb{R}$ *be a non-empty closed interval and* d_Γ *be the distance function to the set* Γ, *i.e., the map* $\mathrm{d}_\Gamma \colon \mathbb{R}_v \to [0, \infty[$ *given, for any* $v \in \mathbb{R}$, *by*

$$\mathrm{d}_\Gamma(v) \doteq \inf_{\bar{v} \in \Gamma} |v - \bar{v}|. \tag{3.5}$$

Then d_Γ *satisfies the following three properties.*

1. *The* inf *in* (3.5) *is a* min, *and* d_Γ *is a convex and 1-Lipschitz continuous function on* $\mathbb{R}$ *with*
$$\mathrm{d}_\Gamma^{-1}(0) = \Gamma. \tag{3.6}$$

2. *For any* $v \in \mathbb{R}$, *the subgradient of* d_Γ *at* v, *i.e., the non-empty subset of* $\mathbb{R}$ *given by*
$$\partial \mathrm{d}_\Gamma(v) = \left\{ \nu \in \mathbb{R} \;\middle|\; \forall\, \hat{v} \in \mathbb{R},\ \mathrm{d}_\Gamma(\hat{v}) \geq \mathrm{d}_\Gamma(v) + \nu(\hat{v} - v) \right\} \tag{3.7}$$
is such that
$$v \notin \Gamma \;\Rightarrow\; \partial \mathrm{d}_\Gamma(v) = \begin{cases} \{1\}, & \text{if } v > \max \Gamma, \\ \{-1\}, & \text{if } v < \min \Gamma. \end{cases} \tag{3.8}$$

3. *The square function* d_Γ^2 *is of class* C^1 *on* $\mathbb{R}$ *with first derivative given, for any* $v \in \mathbb{R}$, *by*
$$\frac{d}{dv}\big(\mathrm{d}_\Gamma^2(v)\big) = \begin{cases} 0, & \text{if } v \in \Gamma, \\ 2\mathrm{d}_\Gamma(v), & \text{if } v > \max \Gamma, \\ -2\mathrm{d}_\Gamma(v), & \text{if } v < \min \Gamma. \end{cases} \tag{3.9}$$

Remark 3.2.1. We can frame what described in Lemma 3.2.2, especially in its multidimensional version, within the concept of *Clarke's generalized gradient* of a scalar function, defined on a region of $\mathbb{R}^d$, which is locally Lipschitz continuous. In this regard, see [36] and Lemma 2.3 in [45, Chap. 3, Sect. 2].

Notation 3.2.1. In light of (3.8), for any $v \in \mathbb{R} \setminus \Gamma$, we identify the set $\partial \mathrm{d}_\Gamma(v)$ with its unique element, i.e., 1 or -1 (a sign). Therefore, in light of (3.6) and (3.9), we simply write

$$\frac{d}{dv}\big(\mathrm{d}_\Gamma^2(\cdot)\big) = 2\mathrm{d}_\Gamma(\cdot)\partial\mathrm{d}_\Gamma(\cdot). \tag{3.10}$$

Remark 3.2.2. As a consequence of Lemma 3.2.2, we emphasize that, for any $v \in \mathbb{R}$ and any sequence $(v_\varepsilon)_\varepsilon$ in $\mathbb{R}$ with $v_\varepsilon \to v$ as $\varepsilon \downarrow 0$, it holds true that

$$\mathrm{d}_\Gamma^2(v_\varepsilon) - \mathrm{d}_\Gamma^2(v) = \big[2\mathrm{d}_\Gamma(v)\partial\mathrm{d}_\Gamma(v) + o_{\varepsilon\downarrow 0}(1)\big](v_\varepsilon - v)$$

or also, by continuity,

$$\mathrm{d}_\Gamma^2(v_\varepsilon) - \mathrm{d}_\Gamma^2(v) = \big[2\mathrm{d}_\Gamma(v_\varepsilon)\partial\mathrm{d}_\Gamma(v_\varepsilon) + \tilde{o}_{\varepsilon\downarrow 0}(1)\big](v_\varepsilon - v)$$

for appropriate infinitesimals (see Notation 3.2.1).

Coming to Theorem 3.1.1, suppose that U is bounded and, without loss of generality, that

$$J(\bar{\boldsymbol{u}}(\cdot); t, x) = 0 \tag{3.11}$$

(see (1.11)). Consider the metric $\mathbf{d}$ on $\mathscr{U}[t,T]$ defined through (1.17) of Definition 1.3.1, whereby

$$(\mathscr{U}[t,T], \mathbf{d})$$

is a complete metric space (Lemma 1.3.2), and the utility functionals

$$J(\cdot\,; t, x), \boldsymbol{J}(\cdot\,; t, x) \colon \mathscr{U}[t,T] \to \mathbb{R},$$

which are continuous w.r.t. the metric $\mathbf{d}$ (Lemma 1.3.3). For any fixed $\varrho \downarrow 0$, we define the *penalty functional*, associated with Problem 2, as the functional $J_\varrho(\cdot\,; t, x) \colon \mathscr{U}[t,T] \to [0, \infty[$ given by

$$J_\varrho(\boldsymbol{u}(\cdot); t, x) \doteq \Big\{\big(J(\boldsymbol{u}(\cdot); t, x) + \varrho\big)^2 + \mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}^2(\boldsymbol{J}(\boldsymbol{u}(\cdot); t, x))\Big\}^{1/2}, \tag{3.12}$$

where $\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}$ is the distance function to the set $\boldsymbol{\Gamma_{t,x}}$ as in (3.5) of Lemma 3.2.2. Then it is simple to realize that $J_\varrho(\cdot\,; t, x)$ satisfies the following three properties.

- $J_\varrho(\cdot\,; t, x)$ is continuous w.r.t. the metric $\mathbf{d}$.
- For $\boldsymbol{u}(\cdot) = \bar{\boldsymbol{u}}(\cdot)$,

$$J_\varrho(\bar{\boldsymbol{u}}(\cdot); t, x) = \varrho \tag{3.13}$$

  (see (3.11) and (3.6) of Lemma 3.2.2). In particular,

$$J_\varrho(\bar{\boldsymbol{u}}(\cdot); t, x) \le \inf_{\boldsymbol{u}(\cdot) \in \mathcal{U}[t,T]} J_\varrho(\boldsymbol{u}(\cdot); t, x) + \varrho.$$

- For any $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$,

$$J_\varrho(\boldsymbol{u}(\cdot);t,x) = 0 \qquad \Longleftrightarrow \qquad \begin{cases} J(\boldsymbol{u}(\cdot);t,x) = -\varrho, \\ \boldsymbol{J}(\boldsymbol{u}(\cdot);t,x) \in \boldsymbol{\Gamma_{t,x}} \end{cases} \tag{3.14}$$

(see (3.6) of Lemma 3.2.2).

Remark 3.2.3. If $J(\,\cdot\,;t,x) \geq 0$, i.e., $\bar{\boldsymbol{u}}(\,\cdot\,)$ is a classical minimum point for $J(\,\cdot\,;t,x)$ (see (3.11)), then $J_\varrho(\,\cdot\,;t,x) \geq \varrho > 0$ (otherwise, we absolutely cannot rely on the possibility of such an inequality).

Therefore, by Lemma 3.2.1 it follows that there exists

$$\bar{\boldsymbol{u}}_\varrho(\cdot) \in \mathscr{U}[t,T]$$

such that the following three conditions hold.

1. $J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \leq \varrho$.
2. $\mathbf{d}(\bar{\boldsymbol{u}}(\cdot), \bar{\boldsymbol{u}}_\varrho(\cdot)) \leq \sqrt{\varrho}$. Therefore, as $\varrho \downarrow 0$,

$$\bar{\boldsymbol{u}}_\varrho(\cdot) \xrightarrow{\mathbf{d}} \bar{\boldsymbol{u}}(\cdot). \tag{3.15}$$

3. For any $\boldsymbol{v}(\cdot) \in \mathscr{U}[t,T]$,

$$J_\varrho(\boldsymbol{v}(\cdot);t,x) - J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \geq -\sqrt{\varrho}\,\mathbf{d}(\boldsymbol{v}(\cdot), \bar{\boldsymbol{u}}_\varrho(\cdot)). \tag{3.16}$$

Fix $\boldsymbol{u}(\cdot) \in \mathscr{U}[t,T]$. For $\varepsilon \in \,]0, T-t[$, let E_t^ε be as in (1.12), i.e.,

$$E_t^\varepsilon = [t, t+\varepsilon]$$

and let $\bar{\boldsymbol{u}}^\varepsilon(\cdot)$ be the spike variation of $\bar{\boldsymbol{u}}(\cdot)$ w.r.t. $\boldsymbol{u}(\cdot)$ and E_t^ε. Then, since

$$\left\{\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot) \neq \bar{\boldsymbol{u}}_\varrho(\cdot)\right\} \subseteq E_t^\varepsilon \times \Omega,$$

we have that

$$\mathbf{d}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot), \bar{\boldsymbol{u}}_\varrho(\cdot)) \leq \varepsilon \tag{3.17}$$

(see (1.17)). Therefore, as $\varepsilon \downarrow 0$,

$$\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot) \xrightarrow{\mathbf{d}} \bar{\boldsymbol{u}}_\varrho(\cdot). \tag{3.18}$$

Consequently, from (3.17) and (3.16), with $\boldsymbol{v}(\cdot) = \bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot)$, we get

$$-\sqrt{\varrho}\,\varepsilon \leq J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x). \tag{3.19}$$

Given the definition of $J_\varrho(\cdot\,;t,x)$ (see (3.12)) and the possibility that (3.14) holds ($J(\cdot\,;t,x)$ can assume negative values), we divide the proof into two cases (unlike what is proposed in [45]).

Case I: For $\varrho \downarrow 0$, $J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \neq 0$ (i.e., > 0). We can rewrite (3.19) as

$$\begin{aligned}
-\sqrt{\varrho}\,\varepsilon &\leq J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \\
&= \frac{J_\varrho^2(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - J_\varrho^2(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \\
&= \frac{\left(J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)+\varrho\right)^2 - \left(J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)+\varrho\right)^2}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \\
&\quad + \frac{\mathrm{d}^2_{\boldsymbol{\Gamma}_{t,x}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)) - \mathrm{d}^2_{\boldsymbol{\Gamma}_{t,x}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)}
\end{aligned} \tag{3.20}$$

by using also the definition (3.12) of $J_\varrho(\cdot\,;t,x)$ (and that, for $a_1, a_2 \in \mathbb{R}$ with $a_1 + a_2 \neq 0$, $a_1 - a_2 = (a_1^2 - a_2^2)/(a_1 + a_2)$). Now, there exist $\psi_\varrho^\varepsilon, \boldsymbol{\psi}_\varrho^\varepsilon \in \mathbb{R}$ with

$$\begin{aligned}
&\frac{\left(J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)+\varrho\right)^2 - \left(J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)+\varrho\right)^2}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \\
&\qquad\qquad = \psi_\varrho^\varepsilon\left[J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\right]
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\frac{\mathrm{d}^2_{\boldsymbol{\Gamma}_{t,x}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)) - \mathrm{d}^2_{\boldsymbol{\Gamma}_{t,x}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \\
&\qquad\qquad = \boldsymbol{\psi}_\varrho^\varepsilon\left[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\right],
\end{aligned} \tag{3.22}$$

and they are respectively

$$\psi_\varrho^\varepsilon = \frac{J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)+\varrho}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} + o^\varrho_{\varepsilon\downarrow 0}(1) \tag{3.23}$$

and

$$\boldsymbol{\psi}_\varrho^\varepsilon = \frac{\mathrm{d}_{\boldsymbol{\Gamma}_{t,x}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma}_{t,x}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} + \boldsymbol{o}^\varrho_{\varepsilon\downarrow 0}(1) \tag{3.24}$$

for appropriate infinitesimals (see also Notation 3.2.1).

Indeed, by continuity of $J(\cdot\,;t,x)$, $\boldsymbol{J}(\cdot\,;t,x)$ and $J_\varrho(\cdot\,;t,x)$ w.r.t. the metric $\mathbf{d}$, together with (3.18), we have

$$J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) = J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + o^{\varrho;1}_{\varepsilon\downarrow 0}(1)$$

and

$$J_\varrho(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) = J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + o^{\varrho,2}_{\varepsilon\downarrow 0}(1),$$

while, by Remark 3.2.2,

$$\begin{aligned}
&\mathrm{d}^2_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x)) - \mathrm{d}^2_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)) = \\
&\qquad \Big[2\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)) \\
&\qquad\qquad + \boldsymbol{o}^{\varrho,1}_{\varepsilon\downarrow 0}(1)\Big]\big[\boldsymbol{J}(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big]
\end{aligned}$$

(for appropriate infinitesimals). Therefore, regarding (3.21),

$$\begin{aligned}
&(J(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x)+\varrho)^2 - (J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)+\varrho)^2 \\
&\quad = \big[J(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) + J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + 2\varrho\big]\big[J(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) - J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big],
\end{aligned}$$

and, from above,

$$\begin{aligned}
\frac{J(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) + J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + 2\varrho}{J_\varrho(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} &= \frac{2J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + 2\varrho + o^{\varrho,1}_{\varepsilon\downarrow 0}(1)}{2J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + o^{\varrho,2}_{\varepsilon\downarrow 0}(1)} \\
&\equiv \frac{J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)+\varrho}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} + o^{\varrho}_{\varepsilon\downarrow 0}(1)
\end{aligned}$$

(which leads to (3.23)), while, regarding (3.22),

$$\begin{aligned}
&\frac{2\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)) + \boldsymbol{o}^{\varrho,1}_{\varepsilon\downarrow 0}(1)}{J_\varrho(\bar{\boldsymbol{u}}^\varepsilon_\varrho(\cdot);t,x) + J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \\
&\qquad = \frac{2\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)) + \boldsymbol{o}^{\varrho,1}_{\varepsilon\downarrow 0}(1)}{2J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + o^{\varrho,2}_{\varepsilon\downarrow 0}(1)} \\
&\qquad\qquad \equiv \frac{\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} + \boldsymbol{o}^{\varrho}_{\varepsilon\downarrow 0}(1)
\end{aligned}$$

(which leads to (3.24)).

For both ψ^ε_ϱ and $\boldsymbol{\psi}^\varepsilon_\varrho$, the numerator of the fraction dependent on ϱ that characterizes them, unless it is infinitesimal for $\varepsilon \downarrow 0$, could have a negative sign (see (3.23) and (3.24)). Nevertheless, with squares,

$$(\psi^\varepsilon_\varrho)^2 + (\boldsymbol{\psi}^\varepsilon_\varrho)^2 = 1 + \boldsymbol{o}^{\varrho,2}_{\varepsilon\downarrow 0}(1)$$

(see (3.12)).

Therefore, for boundedness as $\varepsilon \downarrow 0$ (in $\mathbb{R}$), there exist $\psi_\varrho, \boldsymbol{\psi}_\varrho \in \mathbb{R}$ with

$$(\psi_\varrho)^2 + (\boldsymbol{\psi}_\varrho)^2 = 1$$

and two subsequences, still denoted by $(\psi_\varrho^\varepsilon)_\varepsilon$ and $(\boldsymbol{\psi}_\varrho^\varepsilon)_\varepsilon$, such that, as $\varepsilon \downarrow 0$,

$$\begin{cases} \psi_\varrho^\varepsilon \to \psi_\varrho, \\ \boldsymbol{\psi}_\varrho^\varepsilon \to \boldsymbol{\psi}_\varrho, \end{cases} \tag{3.25}$$

indeed

$$\psi_\varrho = \frac{J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) + \varrho}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \tag{3.26}$$

and

$$\boldsymbol{\psi}_\varrho = \frac{\mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial \mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} \tag{3.27}$$

so, again, there exist $\psi, \boldsymbol{\psi} \in \mathbb{R}$ with

$$\psi^2 + \boldsymbol{\psi}^2 = 1$$

and two subsequences, still denoted by $(\psi_\varrho)_\varrho$ and $(\boldsymbol{\psi}_\varrho)_\varrho$, such that, as $\varrho \downarrow 0$,

$$\begin{cases} \psi_\varrho \to \psi, \\ \boldsymbol{\psi}_\varrho \to \boldsymbol{\psi}. \end{cases} \tag{3.28}$$

In particular, $\psi, \boldsymbol{\psi} \in [-1,1]$, and we get (3.1). Instead, regarding (3.2), we have, for any $\bar{v} \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$, that

$$\begin{aligned} &\boldsymbol{\psi}_\varrho^\varepsilon \big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big] \\ &\quad = \frac{\mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\partial \mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big]}{J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)} + \tilde{\boldsymbol{o}}^\varrho_{\varepsilon\downarrow 0}(1) \end{aligned}$$

(see (3.24)), where, even if $\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \notin \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$,

$$\begin{aligned} &\partial \mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x))\big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big] \\ &\qquad \le \mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\bar{v}) - \mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)) = -\,\mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)) \le 0 \end{aligned}$$

(the first inequality above derives from that of (3.7), where $v \equiv \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)$). Therefore, in any case,

$$\boldsymbol{\psi}_\varrho^\varepsilon \big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big] \le \tilde{\boldsymbol{o}}^\varrho_{\varepsilon\downarrow 0}(1),$$

and, by this, together with (3.25) and (3.28), sending first $\varepsilon \downarrow 0$ and then $\varrho \downarrow 0$,

$$\boldsymbol{\psi}_\varrho\big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x)\big] \le 0,$$

and hence (3.2), as we wanted (see also (3.15)).

Continuing from (3.20) through (3.21) and (3.22), we can write

$$\begin{aligned} -\sqrt{\varrho}\,\varepsilon \le \psi_\varrho^\varepsilon\big[J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot); t, x) - J(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x)\big] \\ + \boldsymbol{\psi}_\varrho^\varepsilon\big[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot); t, x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x)\big]. \end{aligned} \tag{3.29}$$

Let $(p_\varrho(\cdot\,; t), q_\varrho(\cdot\,; t))$, $(P_\varrho(\cdot\,; t), Q_\varrho(\cdot\,; t))$, $\mathcal{H}_\varrho(\cdot\,; t)$, $\delta\mathcal{H}_\varrho(\cdot\,; t, \text{-})$ correspond to the 4-tuple

$$(\bar{\boldsymbol{u}}_\varrho(\cdot), \bar{X}_\varrho(\cdot), \bar{Y}_\varrho(\cdot\,; t), \bar{Z}_\varrho(\cdot\,; t))$$

and $(\boldsymbol{p}_\varrho(\cdot\,; t), \boldsymbol{q}_\varrho(\cdot\,; t))$, $(\boldsymbol{P}_\varrho(\cdot\,; t), \boldsymbol{Q}_\varrho(\cdot\,; t))$, $\boldsymbol{\mathcal{H}}_\varrho(\cdot\,; t)$, $\delta\boldsymbol{\mathcal{H}}_\varrho(\cdot\,; t, \text{-})$ correspond to the 4-tuple

$$(\bar{\boldsymbol{u}}_\varrho(\cdot), \bar{X}_\varrho(\cdot), \bar{\boldsymbol{Y}}_\varrho(\cdot\,; t), \bar{\boldsymbol{Z}}_\varrho(\cdot\,; t))$$

(see Definitions 2.1.2, 2.1.3, 2.1.4, and 2.1.5 and Notation 2.1.2). Then, by Lemma 2.1.1,

$$\liminf_{\varepsilon\downarrow 0} \frac{J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot); t, x) - J(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x)}{\varepsilon} = \mathbf{E}\big[\delta\mathcal{H}_\varrho(t; t, \boldsymbol{u}(t))\big],$$

$$\liminf_{\varepsilon\downarrow 0} \frac{\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot); t, x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x)}{\varepsilon} = \mathbf{E}\big[\delta\boldsymbol{\mathcal{H}}_\varrho(t; t, \boldsymbol{u}(t))\big],$$

and these are actual limits (this will be crucial here). Therefore, by (3.29) and (3.25),

$$-\sqrt{\varrho} \le \mathbf{E}\big[\psi_\varrho\,\delta\mathcal{H}_\varrho(t; t, \boldsymbol{u}(t)) + \boldsymbol{\psi}_\varrho\,\delta\boldsymbol{\mathcal{H}}_\varrho(t; t, \boldsymbol{u}(t))\big],$$

and finally, by (3.15) and (3.28),

$$\mathbf{E}\big[\psi\,\delta\mathcal{H}(t; t, \boldsymbol{u}(t)) + \boldsymbol{\psi}\,\delta\boldsymbol{\mathcal{H}}(t; t, \boldsymbol{u}(t))\big] \ge 0.$$

It is now simple to deduce (3.3), for any $\boldsymbol{u} \in U$ (and $\mathbf{P}$-a.s.), with exactly the same procedure by contradiction that we used in the proof of Theorem 2.1.1.

Case II: For infinitely many $\varrho \downarrow 0$, $J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x) = 0$, i.e.,

$$\begin{cases} J(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x) = -\varrho, \\ \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot); t, x) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}} \end{cases}$$

(see (3.14)). As we have seen above, the key point is to find two sequences $(\psi_\varrho^\varepsilon)_\varepsilon$ and $(\boldsymbol{\psi}_\varrho^\varepsilon)_\varepsilon$ in $\mathbb{R}$, bounded as $\varepsilon \downarrow 0$, such that, as in (3.20)/(3.29),

$$\begin{aligned} J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) &\equiv J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \\ &\le \psi_\varrho^\varepsilon\big[J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - J(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big] \\ &+ \boldsymbol{\psi}_\varrho^\varepsilon\big[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big] \\ &\equiv \psi_\varrho^\varepsilon\big[J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho\big] \\ &+ \boldsymbol{\psi}_\varrho^\varepsilon\big[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big]. \end{aligned} \tag{3.30}$$

Let us analyze this situation by dividing it, in turn, into two further sub-cases w.r.t. the fact that

$$\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}.$$

Sub-case I of Case II: For infinitely many (of such) $\varrho \downarrow 0$,

$$\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \in \operatorname{int}(\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}).$$

Then, corresponding to these values of $\varrho \downarrow 0$, we have also that, as $\varepsilon \downarrow 0$,

$$\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) \in \operatorname{int}(\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}})$$

(see (3.18)), and so, for such parameters,

$$J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) = \big|J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho\big|,$$

since $\operatorname{int}(\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}) \subset \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$ (see (3.12)). Therefore, to get (3.1), (3.2), and (3.30), we can simply take

$$\psi_\varrho^\varepsilon := \operatorname{sgn}(J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho)$$

and

$$\boldsymbol{\psi}_\varrho^\varepsilon := 0,$$

from which, in the limit, $|\psi| = 1$ and $\boldsymbol{\psi} = 0$ (for $v \in \mathbb{R}$, $|v| = \operatorname{sgn}(v)v$ or, also, $\operatorname{sgn}(v)|v| = v$).

Sub-case II of Case II: For infinitely many such $\varrho \downarrow 0$, except at most a finite number, $\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) \in \partial\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$ and, at the same time, as $\varepsilon \downarrow 0$,

$$\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) \notin \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$$

(otherwise, we could proceed as just seen unless we pass to subsequences).

Then, corresponding to these values of $\varrho \downarrow 0$ and $\varepsilon \downarrow 0$,

$$J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) \neq 0$$

(i.e., > 0) and, by (3.12) and equivalently to (3.20) (since $J_\varrho(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x) = 0$),

$$J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) = \frac{\left(J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho\right)^2}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)} + \frac{\mathrm{d}^2_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)}.$$

Again, aiming for (3.30), we look for $\psi_\varrho^\varepsilon, \boldsymbol{\psi}_\varrho^\varepsilon \in \mathbb{R}$ such that, separately,

$$\frac{\left(J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho\right)^2}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)} \le \psi_\varrho^\varepsilon \left[J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho\right] \tag{3.31}$$

and

$$\frac{\mathrm{d}^2_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)} \le \boldsymbol{\psi}_\varrho^\varepsilon \left[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\right]. \tag{3.32}$$

Regarding (3.31), we simply take

$$\psi_\varrho^\varepsilon := \frac{J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) + \varrho}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)},$$

while, regarding (3.32), since

$$\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)) \le \partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))\left[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\right]$$

(see (3.7)), we take

$$\boldsymbol{\psi}_\varrho^\varepsilon := \frac{\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)},$$

and we are done: indeed, on the one hand, regarding (3.1),

$$(\psi_\varrho^\varepsilon)^2 + (\boldsymbol{\psi}_\varrho^\varepsilon)^2 = 1$$

(remember also (3.8)); on the other hand, regarding (3.2), we start from the fact that, for any $\bar{v} \in \boldsymbol{\Gamma_{t,x}}$,

$$\begin{aligned}\boldsymbol{\psi}_\varrho^\varepsilon &\left[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)\right] \\ &= \frac{\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))\left[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)\right]}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)},\end{aligned}$$

where

$$\partial\mathrm{d}_{\boldsymbol{\Gamma_{t,x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))\left[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)\right] < 0,$$

and so we conclude in a perfectly analogous way to what was done in Case I (see also (3.18) and (3.15)).

Remark 3.2.4. Here, we could have made any choice of the type

$$\psi_\varrho^\varepsilon = \frac{J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)+\varrho}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)} + o_{\varepsilon\downarrow 0}^{\varrho}(1)$$

and

$$\boldsymbol{\psi}_\varrho^\varepsilon = \frac{\mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))\partial \mathrm{d}_{\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}}(\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x))}{J_\varrho(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)} + \boldsymbol{o}_{\varepsilon\downarrow 0}^{\varrho}(1)$$

for appropriate infinitesimals not identically zero such that respectively

$$o_{\varepsilon\downarrow 0}^{\varrho}(1)\big[J(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x)+\varrho\big] \geq 0$$

and

$$\boldsymbol{o}_{\varepsilon\downarrow 0}^{\varrho}(1)\big[\boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho^\varepsilon(\cdot);t,x) - \boldsymbol{J}(\bar{\boldsymbol{u}}_\varrho(\cdot);t,x)\big] \geq 0$$

(see (3.31) and (3.32)).

Remark 3.2.5. If the following possible assumption of *local uniqueness* type for the equilibrium 4-tuple $(\bar{\boldsymbol{u}}(\cdot),\bar{X}(\cdot),\bar{Y}(\cdot\,;t),\bar{Z}(\cdot\,;t))$ holds, where $(\bar{\boldsymbol{u}}(\cdot),\bar{X}(\cdot))$ satisfies the state constraint, then Case II of the above proof does not occur (as if we were dealing with a classical minimum point $\bar{\boldsymbol{u}}(\cdot)$ for $J(\cdot\,;t,x)$). See also the second bullet point in Remark 1.2.12.

Assumption. There exists $\bar{\delta} > 0$ such that, for any $\hat{\boldsymbol{u}}(\cdot) = \hat{\boldsymbol{u}}_{\bar{\delta}}(\cdot) \in \mathscr{U}[t,T]$ with $\boldsymbol{J}(\hat{\boldsymbol{u}}(\cdot);t,x) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$ and $\mathbf{d}(\bar{\boldsymbol{u}}(\cdot),\hat{\boldsymbol{u}}(\cdot)) < \bar{\delta}$, if $J(\hat{\boldsymbol{u}}(\cdot);t,x) \leq o_{\bar{\delta}\downarrow 0}(1)$ (for an appropriate infinitesimal), then $\hat{\boldsymbol{u}}(\cdot) = \bar{\boldsymbol{u}}(\cdot)$.

Indeed, we could take $\varrho \in \,]0,\bar{\delta}^2[$ and $\hat{\boldsymbol{u}}(\cdot) = \bar{\boldsymbol{u}}_\varrho(\cdot)$ (remember also (3.11)).

3.3 Unbounded control domain

Regarding the case where U is not necessarily bounded, we were inspired by the techniques in, e.g., [39] and [41] to obtain the following corollary of Theorem 3.1.1.

Corollary 3.3.1 (Maximum principle). *Let* $(\bar{\boldsymbol{u}}(\cdot),\bar{X}(\cdot),\bar{Y}(\cdot\,;t),\bar{Z}(\cdot\,;t))$ *be an equilibrium 4-tuple such that* $(\bar{\boldsymbol{u}}(\cdot),\bar{X}(\cdot))$ *satisfies the state constraint, i.e.,*

$$\boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot);t,x) \equiv \bar{\boldsymbol{Y}}(t;t) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}},$$

and suppose that

$$\bar{\boldsymbol{u}}(\cdot) \in \mathcal{L}_{\mathbb{F}}^{\infty}(t,T;\mathbb{R}^n).$$

Then there exist $\psi,\boldsymbol{\psi} \in [-1,1]$ *with* $\psi^2+\boldsymbol{\psi}^2 = 1$ *such that the following two conditions hold.*

1. *For any* $\bar{v} \in \mathbf{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$, $\boldsymbol{\psi}\big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot);t,x)\big] \leq 0$.
2. *Let* $(p(\cdot\,;t), q(\cdot\,;t))$, $(P(\cdot\,;t), Q(\cdot\,;t))$, $\mathcal{H}(\cdot\,;t)$, $\delta\mathcal{H}(\cdot\,;t,\text{-})$ *correspond to the 4-tuple*
$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$$
and $(\boldsymbol{p}(\cdot\,;t), \boldsymbol{q}(\cdot\,;t))$, $(\boldsymbol{P}(\cdot\,;t), \boldsymbol{Q}(\cdot\,;t))$, $\boldsymbol{\mathcal{H}}(\cdot\,;t)$, $\delta\boldsymbol{\mathcal{H}}(\cdot\,;t,\text{-})$ *correspond to the 4-tuple*
$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{\boldsymbol{Y}}(\cdot\,;t), \bar{\boldsymbol{Z}}(\cdot\,;t)).$$
Then, for any $\boldsymbol{u} \in U$ *(and* $\mathbf{P}$*-a.s.),*
$$\psi\,\delta\mathcal{H}(t;t,\boldsymbol{u}) + \boldsymbol{\psi}\,\delta\boldsymbol{\mathcal{H}}(t;t,\boldsymbol{u}) \geq 0.$$

Proof. We denote by $\bar{\Omega}$ the subset of Ω consisting of $\omega \in \Omega$ for which the process $\bar{\boldsymbol{u}}(\cdot)$ is well defined ($\mathbf{P}[\bar{\Omega}] = 1$). For $(s,\omega) \in [t,T] \times \bar{\Omega}$ and $j \in \mathbb{N}$, set
$$U_j(s,\omega) \coloneqq \{\, \boldsymbol{u} \in U \mid |\boldsymbol{u}| \leq |\bar{\boldsymbol{u}}(s,\omega)| + j \,\}\,.$$
(that is, the intersection of U and the closed sphere in $\mathbb{R}^n$ of center the origin and of radius $|\bar{\boldsymbol{u}}(s,\omega)| + j$). We point out the following.

- For any $(s,\omega) \in [t,T] \times \bar{\Omega}$, $\bar{\boldsymbol{u}}(s,\omega) \in U_0(s,\omega)$.
- For any $(s,\omega) \in [t,T] \times \bar{\Omega}$ and $j \in \mathbb{N}$, $U_j(s,\omega) \subseteq U_{j+1}(s,\omega)$.
- For any $(s,\omega) \in [t,T] \times \bar{\Omega}$, $\bigcup_{j=0}^{\infty} U_j(s,\omega) = U$.

For $j \in \mathbb{N}$, set
$$\mathscr{U}_j[t,T] \coloneqq \big\{\, \boldsymbol{u}(\cdot) \in \mathcal{L}^p_{\mathbb{F}}(t,T;\mathbb{R}^n) \mid \forall\, (s,\omega) \in [t,T] \times \bar{\Omega}, \boldsymbol{u}(s,\omega) \in U_j(s,\omega) \,\big\}\,.$$

We point out the following.

- $\bar{\boldsymbol{u}}(\cdot) \in \mathscr{U}_0[t,T]$.
- For any $j \in \mathbb{N}$, $\mathscr{U}_j[t,T] \subseteq \mathscr{U}_{j+1}[t,T]$.
- $\bigcup_{j=0}^{\infty} \mathscr{U}_j[t,T] = \mathscr{U}[t,T]$.

Fix $j \in \mathbb{N}$. For any $\boldsymbol{u}(\cdot) \in \mathscr{U}_j[t,T]$, we have that, for any $(s,\omega) \in [t,T] \times \bar{\Omega}$,
$$|\boldsymbol{u}(s,\omega)| \leq \|\bar{\boldsymbol{u}}(\cdot)\|_\infty + j.$$
In particular, each element of $\mathscr{U}_j[t,T]$ assumes values in the intersection of U and the closed sphere in $\mathbb{R}^n$ of center the origin and of radius $\|\bar{\boldsymbol{u}}(\cdot)\|_\infty + j$, which we denote with
$$\bar{U}_j.$$

Therefore, $\bar{U}_j$ takes the place of a bounded control domain (subset of U) and so, by virtue of Theorem 3.1.1 where $\mathscr{U}[t,T]$ is replaced by $\mathscr{U}_j[t,T]$ (since $\bar{\boldsymbol{u}}(\cdot) \in \mathscr{U}_j[t,T]$), we can assert that there exist $\psi_j, \boldsymbol{\psi}_j \in [-1,1]$ with $\psi_j^2 + \boldsymbol{\psi}_j^2 = 1$ such that the following two conditions hold.

1. For any $\bar{v} \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$, $\boldsymbol{\psi}_j\big[\bar{v} - \boldsymbol{J}(\bar{\boldsymbol{u}}(\cdot); t, x)\big] \leq 0$.
2. Let $(p(\cdot\,;t), q(\cdot\,;t))$, $(P(\cdot\,;t), Q(\cdot\,;t))$, $\mathcal{H}(\cdot\,;t)$, $\delta\mathcal{H}(\cdot\,;t,\text{-})$ correspond to the 4-tuple
$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$$
and $(\boldsymbol{p}(\cdot\,;t), \boldsymbol{q}(\cdot\,;t))$, $(\boldsymbol{P}(\cdot\,;t), \boldsymbol{Q}(\cdot\,;t))$, $\boldsymbol{\mathcal{H}}(\cdot\,;t)$, $\delta\boldsymbol{\mathcal{H}}(\cdot\,;t,\text{-})$ correspond to the 4-tuple
$$(\bar{\boldsymbol{u}}(\cdot), \bar{X}(\cdot), \bar{\boldsymbol{Y}}(\cdot\,;t), \bar{\boldsymbol{Z}}(\cdot\,;t)).$$
Then, for any $\boldsymbol{u} \in \bar{U}_j$ (and $\mathbf{P}$-a.s.),
$$\psi_j\, \delta\mathcal{H}(t;t,\boldsymbol{u}) + \boldsymbol{\psi}_j\, \delta\boldsymbol{\mathcal{H}}(t;t,\boldsymbol{u}) \geq 0.$$

(The maps $\mathcal{H}(\cdot\,;t)$ and $\boldsymbol{\mathcal{H}}(\cdot\,;t)$, defined also on U, are restricted w.r.t. the variable $\boldsymbol{u}$ to a subset of U itself). Now the thesis follows easily by taking convergent subsequences of $(\psi_j)_j$ and $(\boldsymbol{\psi}_j)_j$ (since $\bigcup_{j=0}^{\infty} \bar{U}_j = U$). □

3.4 Application to portfolio management (II)

Continuing from Section 2.4, we also take another utility function $\hat{\boldsymbol{v}}(\cdot)$, two other discount functions $\boldsymbol{\hbar}(\cdot\,;t), \hat{\boldsymbol{\hbar}}(\cdot\,;t)$ on $[t,T]$ and $\boldsymbol{\beta}(\cdot\,;t) \in \mathcal{L}_{\mathbb{F}}^{\infty}(t,T;\mathbb{R})$ positive. Hence, as a recursive utility system of type (1.14), we propose

$$\begin{cases} d\boldsymbol{Y}(s;t) = \boldsymbol{\hbar}(s;t)\boldsymbol{\beta}(s;t)\boldsymbol{Y}(s;t)ds + \boldsymbol{Z}(s;t)dW(s), \\ \boldsymbol{Y}(T;t) = -\hat{\boldsymbol{\hbar}}(T;t)\hat{\boldsymbol{v}}(X(T)), \end{cases}$$

(where $s \in [t,T]$). An economic justification for such a system may be that, under suitable assumptions, the state constraint associated with this BSDE through the utility domain $\boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$, i.e.,

$$\boldsymbol{Y}(t;t) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$$

(see Definition 1.2.9) becomes a risk/acceptability constraint: that is, the additional recursive utility $\boldsymbol{Y}(\cdot\,;t)$ derives from an appropriate dynamic risk measure defined by means of a g-expectation (see, among others, [37]).

Therefore, Theorem 3.1.1 becomes the following (see Chapter 3).

Theorem 3.4.1. *Let* $((\bar{\zeta}(\cdot), \bar{c}(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$ *be an equilibrium 4-tuple such that*

$$\boldsymbol{J}((\bar{\zeta}(\cdot), \bar{c}(\cdot)); t, x) \equiv \bar{\boldsymbol{Y}}(t;t) \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}.$$

Then there exist $\psi, \boldsymbol{\psi} \in [-1, 1]$ *with* $\psi^2 + \boldsymbol{\psi}^2 = 1$ *such that the following two conditions hold.*

1. *For any* $\bar{v} \in \boldsymbol{\Gamma}_{\boldsymbol{t},\boldsymbol{x}}$, $\boldsymbol{\psi}\big[\bar{v} - \boldsymbol{J}((\bar{\zeta}(\cdot), \bar{c}(\cdot)); t, x)\big] \leq 0$.
2. *Let* $(p(\cdot\,;t), q(\cdot\,;t))$ *and* $(P(\cdot\,;t), Q(\cdot\,;t))$ *correspond to the 4-tuple*

$$((\bar{\zeta}(\cdot), \bar{c}(\cdot)), \bar{X}(\cdot), \bar{Y}(\cdot\,;t), \bar{Z}(\cdot\,;t))$$

and $(\boldsymbol{p}(\cdot\,;t), \boldsymbol{q}(\cdot\,;t))$ *and* $(\boldsymbol{P}(\cdot\,;t), \boldsymbol{Q}(\cdot\,;t))$ *correspond to the 4-tuple*

$$((\bar{\zeta}(\cdot), \bar{c}(\cdot)), \bar{X}(\cdot), \bar{\boldsymbol{Y}}(\cdot\,;t), \bar{\boldsymbol{Z}}(\cdot\,;t)).$$

Then, for any $(\zeta, c) \in [-1, 1] \times [0, 1]$ *(and* $\mathbf{P}$*-a.s.),*

$$\begin{aligned}
&\big[\psi p(t;t) + \boldsymbol{\psi}\boldsymbol{p}(t;t)\big]\big[\mu(\zeta - \bar{\zeta}(t,x)) - (c - \bar{c}(t,x))\big] \\
&+ \sigma\big[\psi q(t;t) + \boldsymbol{\psi}\boldsymbol{q}(t;t)\big]\big[\zeta - \bar{\zeta}(t,x)\big] + \tfrac{x}{2}\sigma^2\big[\psi P(t;t) + \boldsymbol{\psi}\boldsymbol{P}(t;t)\big]\big[\zeta - \bar{\zeta}(t,x)\big]^2 \\
&\qquad - \psi\Big\{\tfrac{1}{x}\big[\upsilon(xc) - \upsilon(x\bar{c}(t,x))\big] + \sigma p(t;t)\gamma(t;t)\big[\zeta - \bar{\zeta}(t,x)\big]\Big\} \geq 0.
\end{aligned}$$

Remark 3.4.1. Again, generalization to the case of multiple stocks would basically be a matter of formalization.

Conclusions

We have formulated two classes of time-inconsistent recursive stochastic optimal control problems, namely, unconstrained and constrained, where the notion of optimality is defined by means of subgame-perfect equilibrium.

For both of these classes, we have obtained necessary conditions for existence in the form of a maximum principle, which also contains sufficient conditions in the unconstrained case, relying on a generalized second-order Hamiltonian function.

Under suitable conditions of analytical-geometric regularity, it is possible to restrict attention to the first-order part of the Hamiltonian alone, as is done with the investment-consumption policies considered in Section 2.4.

With regard to possible future developments of the approach discussed in this Thesis, we highlight the following.

As far as the state constraint is concerned, it should be possible to use a constraint on an expected value that is not derived from an admissible recursive utility, or even an infinite-dimensional constraint such as

$$X(T) \in \mathcal{Q} \subset L^2_T(\Omega; \mathbb{R})$$

(see, e.g., [42], [16], and [46]).

In the context of Theorem 3.1.1 (and Corollary 3.3.1), it might be possible to obtain a "unique" generalized Hamiltonian such that (3.3) could be written in a compact way, perhaps keeping implicit the two multipliers $\psi, \boldsymbol{\psi}$.

In the search for a concrete equilibrium policy $\mathbf{\Pi}$, in a practical situation such as that discussed in Section 2.4, the Hamilton–Jacobi–Bellman equation associated with the problem could be set up with an appropriate *ansatz* for the value function (see, e.g., [14] and [13]).

Also with regard to practical applications, other portfolio management problems should be explored, with different choices of recursive utility and constraint.

Even a completely new theory could be constructed once the utility functional $J(\,\cdot\,;t,x)$ has been modified as, among others,

$$J(\boldsymbol{u}(\cdot);t,x) = \mathbf{E}\left[\int_t^T \ell(s,X(s),\boldsymbol{u}(s),Y(s;t),Z(s;t);t)\,ds \right.$$
$$\left. + \varphi(X(T);t) + \gamma(Y(t;t);t)\right]$$

(see, e.g., [27]).

Finally, extensions to the infinite horizon case

$$T = \infty$$

or to random horizons $\tau(\cdot)$ (stopping times), should be investigated as well.

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