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PhD in Computer Science and Mathematics of Calculus

REPORT OF AN INTRODUCTION TO REGULARITY STRUCTURES

A Regularity Structure for Rough Volatility

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Hölder distributions

Consider $d \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and $\mathbb{R}^d = \mathbb{R}^d_x$, $x = (x_1, \dots, x_d)$, equipped by the Euclidean topology and the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ with the Lebesgue measure on it. We denote by $\mathcal{K}_d$ the subclass of $\mathcal{B}(\mathbb{R}^d)$ consisting of the compact subsets K of $\mathbb{R}^d$ and, for $\rho \in \mathbb{R}^*_+ =]0, \infty[$ and $x \in \mathbb{R}^d$, by $B_\rho(x)$ the open ball in $\mathbb{R}^d$ of radius ρ and center x. Whenever $x = 0$, we write $B_\rho \coloneqq B_\rho(0)$.

A *smooth test* or *bump function* on $\mathbb{R}^d$ is a function $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ which is infinitely differentiable and compactly supported. The space of the test functions on $\mathbb{R}^d$ is denoted by $\mathcal{D}_d \coloneqq \mathcal{D}(\mathbb{R}^d) \doteq C^\infty_c(\mathbb{R}^d; \mathbb{R})$ and it's associated to the topology induced by the following uniform convergence notion: given $\varphi, (\varphi_n)_n$ in $\mathcal{D}_d$, $\varphi_n \to \varphi$ in $\mathcal{D}_d$ as $n \to \infty$ if there exists $K \in \mathcal{K}_d$ with $\operatorname{supp} \varphi \cup \operatorname{supp} \varphi_n \subseteq K$ for n large enough and, for any multi-index $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, $\|\mathrm{D}^k \varphi - \mathrm{D}^k \varphi_n\|_\infty \to 0$ as $n \to \infty$, where $\mathrm{D}^k = \partial^{k_1}_{x_1} \cdots \partial^{k_d}_{x_d}$ and $\|\cdot\|_\infty$ is the usual uniform norm. Thereby, $\mathcal{D}_d$ is a complete and locally convex topological not metrizable (real) vector space satisfying the so-called Heine-Borel property. For $r \in \mathbb{N}^*$, we denote

$$\mathcal{D}_{d,r} \coloneqq \{ \, \varphi \in \mathcal{D}_d \mid \operatorname{supp} \varphi \subset B_r \text{ and } \|\varphi\|_{C^r} \le 1 \, \}$$

where $\|\cdot\|_{C^r} = \max_{k \in \mathbb{N}^d, |k| \le r} \|\mathrm{D}^k \cdot\|_\infty$ and $|k| = k_1 + \cdots + k_d$.

A *Schwartz distribution* or *generalized function* on $\mathbb{R}^d$ is a function $\eta \colon \mathcal{D}_d \to \mathbb{R}$ which is linear and (sequentially) continuous w.r.t. the above topology on $\mathcal{D}_d$. The space of the distributions on $\mathbb{R}^d$ is therefore the dual space of $\mathcal{D}_d$, in symbols $\mathcal{D}'_d \coloneqq (\mathcal{D}_d)'$, and as such it's paired with the weak-star topology induced by this pointwise convergence notion: given $\eta, (\eta_n)_n$ in $\mathcal{D}'_d$, $\eta_n \to \eta$ in $\mathcal{D}'_d$ as $n \to \infty$ if, for any $\varphi \in \mathcal{D}_d$, $\eta_n(\varphi) \to \eta(\varphi)$ (in $\mathbb{R}$) as $n \to \infty$. Hence, $\mathcal{D}'_d$ is a locally convex topological not metrizable vector space. For $\eta \in \mathcal{D}'_d$ and $\varphi \in \mathcal{D}_d$, we write $\langle \eta \,; \varphi \rangle \coloneqq \eta(\varphi)$.

Every function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R})$ is identified with the distribution (on $\mathbb{R}^d$) defined by $\varphi \mapsto \int_{\mathbb{R}^d} f \varphi \, \mathrm{d}x$, $\varphi \in \mathcal{D}_d$, and w.r.t. that standpoint $\mathcal{D}_d$ results a weak-star dense subset of $\mathcal{D}'_d$. Even every Radon (tight) measure μ on $\mathcal{B}(\mathbb{R}^d)$ is viewed as a distribution, that given by $\varphi \mapsto \int_{\mathbb{R}^d} \varphi \, \mu(\mathrm{d}x)$, $\varphi \in \mathcal{D}_d$.

For $\eta \in \mathcal{D}'_d$ and $k \in \mathbb{N}^d$, the *(distributional) derivative* $\mathrm{D}^k \eta \in \mathcal{D}'_d$ *of order* k of η takes values

$$\langle \mathrm{D}^k \eta \,; \varphi \rangle \doteq (-1)^{|k|} \langle \eta \,; \mathrm{D}^k \varphi \rangle, \quad \varphi \in \mathcal{D}_d.$$

For $\varphi \in \mathcal{D}_d$, $\lambda \in]0, 1]$ and $x \in \mathbb{R}^d$, the *scaled function* $\varphi^\lambda_x \in \mathcal{D}_d$ is constructed from φ reducing it by factor λ and centering it at point x without changing its integral on $\mathbb{R}^d$:

$$\varphi^\lambda_x(\,\cdot\,) \doteq \lambda^{-d} \varphi\big(\lambda^{-1}(\,\cdot - x)\big).$$

Whenever $x = 0$, we write $\varphi^\lambda \coloneqq \varphi^\lambda_0$. Note that, if $\varphi \in \mathcal{D}_{d,r}$ for $r \in \mathbb{N}^*$, then $\operatorname{supp} \varphi^\lambda_x \subset B_{\lambda r}(x)$ and $\|\varphi^\lambda_x\|_{C^r} \le \lambda^{-r}$. Moreover, for any $k \in \mathbb{N}^d$, $\mathrm{D}^k \varphi^\lambda_x = \lambda^{-|k|} (\mathrm{D}^k \varphi)^\lambda_x$.

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For $\alpha \in \mathbb{R}_+ = [0,\infty[$, if $\alpha \notin \mathbb{N}$ then, leaning on the concept of (locally) Hölder continuous function, the space of *α-Hölder functions* on $\mathbb{R}^d$ is defined as

$$\mathcal{C}_d^\alpha \coloneqq \mathcal{C}^\alpha(\mathbb{R}^d;\mathbb{R}) \doteq \left\{ f \in C^{\lfloor\alpha\rfloor}(\mathbb{R}^d;\mathbb{R}) \,\middle|\, \forall\, k \in \mathbb{N}^d \text{ with } |k| = \lfloor\alpha\rfloor,\ \mathrm{D}^k f \in \mathcal{C}^{\{\alpha\}}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}) \right\}$$

whereas, for $\alpha \in \mathbb{R}_- =]-\infty,0]$, if $\alpha \notin \mathbb{Z}$ then the space of *α-Hölder distributions* on $\mathbb{R}^d$ is defined as

$$\mathcal{C}_d^\alpha \coloneqq \left\{ \eta \in \mathcal{D}'_d \,\middle|\, \forall\, K \in \mathcal{K}_d,\ \exists\, C_K \in \mathbb{R}_+^* : \forall\, \varphi \in \mathcal{D}_{d,\lceil-\alpha\rceil}, \lambda \in]0,1] \text{ and } x \in K,\ |\langle \eta\,;\varphi_x^\lambda\rangle| \le C_K \lambda^\alpha \right\}$$

and finally, for $r \in \mathbb{N}^*$, the space of *Hölder distributions of finite order r* on $\mathbb{R}^d$ is defined as

$$\mathcal{C}_d^{-r} \coloneqq \left\{ \eta \in \mathcal{D}'_d \,\middle|\, r \text{ is the min of } s \in \mathbb{N}^* \text{ s.t.},\ \forall\, K \in \mathcal{K}_d,\ \exists\, C_K \in \mathbb{R}_+^* : \forall\, \varphi \in \mathcal{D}_d,\ |\langle \eta\,;\varphi\rangle| \le C_K \|\varphi\|_{C^s} \right\}.$$

Every distribution in $\mathcal{C}_d^{-r}$ is canonically definable on $C_c^r(\mathbb{R}^d;\mathbb{R})$ (through a continuity standard argument). Observe that, for any $\alpha,\beta \in \mathbb{R}\setminus\mathbb{N}$, if $\alpha < \beta$ then $\mathcal{C}_d^\beta \subset \mathcal{C}_d^\alpha$. Moreover, for any $k \in \mathbb{N}^d$, $\mathrm{D}^k(\mathcal{C}_d^\alpha) \subset \mathcal{C}_d^{\alpha-|k|}$.

Remember the main issues about the operation of product between distributions, summarized by the Schwartz impossibility theorem, and yet the importance of Hölder distributions regarding that.

As far as $d = 1$, we omit the subscript d from each of the previously defined spaces.

Regular wavelets

For $n \in \mathbb{N}$, we denote by $\mathbb{R}^d[x;n]$ the class of the real polynomials on $\mathbb{R}^d = \mathbb{R}^d_x$ of degree n.

Theorem (Daubechies, '88)**.** *Given $r \in \mathbb{N}^*$, there exists $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ with the following properties.*

1. *The function φ is of class C_b^r and has compact support.*

2. *For every $P \in \mathbb{R}^d[x;r]$, there exists $\hat{P} \in \mathbb{R}^d[x;r]$ such for which $P(\,\cdot\,) = \sum_{h\in\mathbb{Z}^d} \hat{P}(h)\varphi(\cdot - h)$.*

3. *For any $h \in \mathbb{Z}^d$, $\int_{\mathbb{R}^d} \varphi(x)\varphi(x-h)\,\mathrm{d}x = \delta_{h,0}$.*

4. *There exists a sequence $(a_h)_{h\in\mathbb{Z}^d}$ in $\mathbb{R}$ such that, for any $x \in \mathbb{R}^d$, $2^{-d/2}\varphi(x/2) = \sum_{h\in\mathbb{Z}^d} a_h \varphi(x-h)$.*

The existence of such a function φ is equivalent to the existence of a wavelet basis of $L_d^2 = L^2(\mathbb{R}^d;\mathbb{R})$ consisting of $\|\cdot\|_{L_d^2}$-orthonormal C_b^r functions with compact support (proceeding according to the wavelet standard analysis of Meyer, '92). Indeed, given $r \in \mathbb{N}^*$ and taken φ as in the previous theorem, for $n \in \mathbb{N}$ we set $\Lambda^n \coloneqq 2^{-n}\mathbb{Z}^d$ and, for any $\psi \in C_c^r(\mathbb{R}^d;\mathbb{R})$ and $y \in \Lambda^n$, $\psi_y^n(\,\cdot\,) \coloneqq 2^{nd/2}\psi(2^n(\cdot - y))$. Then there exists $\Phi \subset C_c^r(\mathbb{R}^d;\mathbb{R})$ with $|\Phi| = \#\Phi < \infty$ which is orthogonal to $\mathbb{R}^d[x;r]$ and such that $\{\varphi_y^0\}_{y\in\mathbb{Z}^d} \cup \{\hat{\varphi}_y^n\}_{\hat{\varphi}\in\Phi, n\in\mathbb{N}, y\in\Lambda^n}$ constitutes an orthonormal basis of L_d^2 (intimately related to $\mathcal{C}_d^{-r}\ldots$).

Regularity structures

A *regularity structure* is a triple $\mathscr{T} = (A,T,G)$ made by the following three elements.

- An *index set* A: a subset of $\mathbb{R}$ with $0 \in A$ which is bounded from below and locally finite.

- A *model space* T: a graded vector space indexed over A of finite-dimensional vector spaces T_α, $\alpha \in A$, each of which admits basis of *symbols* $\{\boldsymbol{\tau}_{\alpha,i}\}_{i\in I_\alpha}$, $|I_\alpha| < \infty$, i.e. $T = \bigoplus_{\alpha\in A} T_\alpha = \bigoplus_{\alpha\in A} \langle \boldsymbol{\tau}_{\alpha,i} \,|\, i \in I_\alpha\rangle$, where $T_0 = \langle \mathbf{1}\rangle \cong \mathbb{R}$. Elements $\boldsymbol{\tau}_\alpha \in T_\alpha$, $\alpha \in A$, are said to have *homogeneity* or *degree* $|\boldsymbol{\tau}_\alpha|$ equal to α. Given $\boldsymbol{\tau} \in T$ and $\alpha \in A$, we write $\|\boldsymbol{\tau}\|_\alpha$ for a chosen norm $\|\cdot\|$ of its component in T_α.

- A *structure group* G: a $\circ$-group of linear operators Γ acting on T with $\Gamma\mathbf{1} = \mathbf{1}$ which satisfy a nilpotency property in the meaning that, for any $\alpha \in A$ and $\boldsymbol{\tau}_\alpha \in T_\alpha$,

$$\Gamma\boldsymbol{\tau}_\alpha - \boldsymbol{\tau}_\alpha \in T_{<\alpha} \doteq \bigoplus_{\alpha'<\alpha} T_{\alpha'}.$$

A *model for* $\mathscr{T}$ on $\mathbb{R}^d$ is then a pair $\mathrm{M} = (\Pi, \Gamma)$ composed of the following two families.

- A map $\Gamma = (\Gamma_{x,y})_{x,y\in\mathbb{R}^d} \colon \mathbb{R}^d \times \mathbb{R}^d \to G$ such that, for any $x, y, z \in \mathbb{R}^d$, $\Gamma_{x,y} \circ \Gamma_{y,z} = \Gamma_{x,z}$.

- A map $\Pi = (\Pi_x)_{x\in\mathbb{R}^d} \colon \mathbb{R}^d \to \mathcal{L}(T; \mathcal{D}'_d)$ such that, for any $x, y \in \mathbb{R}^d$, $\Pi_x \circ \Gamma_{x,y} = \Pi_y$ and furthermore, letting r be the smallest integer $> |\min A|$ then, for any $\gamma \in \mathbb{R}^*_+$ and $K \in \mathcal{K}_d$, there exists $C = C_{\gamma,K} \in \mathbb{R}^*_+$ such that, by varying $\alpha \in A$ with $\alpha \le \gamma$, $\boldsymbol{\tau}_\alpha \in T_\alpha$, $\varphi \in \mathcal{D}_{d,r}$, $\lambda \in]0, 1]$, $x, y \in K$ and $\alpha' < \alpha$,

$$|(\Pi_x \boldsymbol{\tau}_\alpha)(\varphi_x^\lambda)| \le C\lambda^\alpha \|\boldsymbol{\tau}_\alpha\| \quad \text{and} \quad \|\Gamma_{x,y}\boldsymbol{\tau}_\alpha\|_{\alpha'} \le C|x-y|^{\alpha-\alpha'}\|\boldsymbol{\tau}_\alpha\|.$$

We say that Π_x, $x \in \mathbb{R}^d$, *realizes* an element of T as a distribution on $\mathbb{R}^d$.

Skorohod integral

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space such that there exists a 1D standard Brownian motion $W = (W(t))_{t\ge 0}$ on it, provided with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ generated by W, and fix also $T \in \mathbb{R}^*_+$.

For any $n \in \mathbb{N}^*$, the *n-fold iterated* or *multiple Itô integral* (w.r.t. W) of a symmetric function f in $L^2_{n,T} \coloneqq L^2([0,T]^n; \mathbb{R})$ is the random variable in $L^2(\mathbf{P}) \coloneqq L^2((\Omega, \mathcal{F}, \mathbf{P}); \mathbb{R})$ computed as

$$I_n(f) \doteq \int_{[0,T]^n} f(t_1, \dots, t_n)\, \mathrm{d}W(t_1) \cdots \mathrm{d}W(t_n).$$

We remark that $I_n(f)$ is $\mathcal{F}_T$-measurable with $\mathbf{E}[I_n(f)] = 0$ and $\mathbf{E}[I_n(f)^2] = n!\|f\|^2_{L^2_{n,T}}$.

Theorem (Wiener-Itô chaos expansion)**.** *Let $X \in L^2(\mathbf{P})$ be $\mathcal{F}_T$-measurable. Then there exists an (essentially) unique sequence $(f_n)_{n\in\mathbb{N}^*}$ of symmetric functions f_n in $L^2_{n,T}$ such that, as limit in $L^2(\mathbf{P})$,*

$$X = \mathbf{E}[X] + \sum_{n=1}^{\infty} I_n(f_n).$$

Furthermore, $\mathbf{E}[X^2] = \mathbf{E}[X]^2 + \sum_{n=1}^\infty n!\|f_n\|^2_{L^2_{n,T}}$.

Now let $u = (u(t))_{t=0}^T$ be a 1D (stochastic) process such that, for any $t \in [0,T]$, $u(t) \in L^2(\mathbf{P})$ is $\mathcal{F}_T$-measurable, and therefore let $(f_n(\,\cdot\,, t))_{n\in\mathbb{N}^*}$ be the sequence of symmetric functions $f_n(\,\cdot\,, t)$ in $L^2_{n,T}$ which determine the Wiener-Itô chaos expansion of $u(t)$. Consider, for any $n \in \mathbb{N}^*$, the symmetrization $\widetilde{f}_n \in L^2_{n+1,T}$ of $f_n(\,\cdot\,, t)$ as a $L^2_{n+1,T}$-function defined on $[0,T]^n \times [0,T]_t$: for $t_1, \dots, t_n, t \in [0,T]$,

$$\widetilde{f}_n(t_1, \dots, t_n, t) = \frac{1}{n+1}\Bigg\{ f_n(t_1, \dots, t_n, t) + \sum_{j=1}^{n} f_n(t_1, \dots, t_{j-1}, t, t_{j+1}, \dots, t_n, t_j) \Bigg\}.$$

Let's assume also that $\mathbf{E}\big[\int_0^T u^2(t)\,\mathrm{d}t\big] < \infty$. Then u is *Skorohod integrable* (w.r.t. W) if the series

$$\int_0^T u(t)\, \delta W(t) \coloneqq \delta(u) \doteq \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n)$$

converges in $L^2(\mathbf{P})$ and, in such a case, $u \in \operatorname{Dom}\delta$ and $\delta(u)$ is the *Skorohod integral* of u (w.r.t. W).

Seeing δ as an operator from $\operatorname{Dom}\delta$ into $L^2(\mathbf{P})$, called the *divergence operator*, δ results an unbounded and closed linear operator with, for $u \in \operatorname{Dom}\delta$, $\mathbf{E}[\delta(u)] = 0$ and $\mathbf{E}[\delta(u)^2] = \sum_{n=0}^\infty (n+1)!\|\widetilde{f}_n\|^2_{L^2_{n+1,T}}$.

Theorem. *Let $u = (u(t))_{t=0}^T$ be a 1D process which is $(\mathcal{F}_t)_{t=0}^T$-adapted with $\mathbf{E}\big[\int_0^T u^2(t)\,\mathrm{d}t\big] < \infty$. Then u is Skorohod integrable and its Skorohod integral coincides with its Itô integral (w.r.t. W):*

$$\delta(u) = \int_0^T u(t)\, \mathrm{d}W(t).$$

Keep in mind that in general, for arbitrary $u \in \operatorname{Dom}\delta$ and $\zeta\colon \Omega \to \mathbb{R}$ random variable which is $\mathcal{F}_T$-measurable and such that $\zeta u := (\zeta u(t))_{t=0}^T \in \operatorname{Dom}\delta$, then $\delta(\zeta u) \neq \zeta\delta(u)$.

Wick product

A *rapidly decreasing (smooth) function* on $\mathbb{R}^d$ is a function $\varphi \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that, for any $n \in \mathbb{N}$ and $k \in \mathbb{N}^d$, $\|\varphi\|_{n,k} \doteq \sup_{x\in\mathbb{R}^d}|x|^n|\mathrm{D}^k\varphi(x)| < \infty$. The space of the rapidly decreasing functions on $\mathbb{R}^d$ is the *Schwartz space* (w.r.t. d), is denoted by $\mathcal{S}_d := \mathcal{S}(\mathbb{R}^d) \subset L^2_d$ and is flanked by the topology induced by the countable family of seminorms $\|\cdot\|_{n,k}$ which makes it a Fréchet space (so complete and locally convex T_2 topological metrizable vector space). A *tempered* or *slowly increasing distribution* on $\mathbb{R}^d$ is a function from $\mathcal{S}_d$ into $\mathbb{R}$ which is linear and (sequentially) continuous w.r.t. that topology on $\mathcal{S}_d$. The space of the tempered distributions on $\mathbb{R}^d$ is then the dual space $\mathcal{S}'_d := (\mathcal{S}_d)' \subset \mathcal{D}'_d$ of $\mathcal{S}_d$ and is coupled with the weak-star topology. As far as $d = 1$, we omit the subscript d from each of the spaces above.

Theorem (Bochner-Minlos-Sazonov)**.** *There exists a complete probability measure* $\mathbf{P}$ *defined on the Borel* σ*-algebra* $\mathcal{F}$ *on* $\Omega := \mathcal{S}'$ *such that, for every* $\varphi \in \mathcal{S}$,

$$\int_\Omega e^{i\langle\omega;\varphi\rangle}\,\mathbf{P}(\mathrm{d}\omega) = \exp\biggl(-\frac{1}{2}\|\varphi\|^2_{L^2}\biggr).$$

We name $\mathbf{P}$ the *white noise probability measure* and $(\Omega,\mathcal{F},\mathbf{P})$ the *white noise probability space*. The *(smoothed) white noise process* is the map $w_\cdot\colon L^2 \to L^2(\mathbf{P})$ identified by placing, for $\varphi \in \mathcal{S}$ and $\omega \in \Omega$,

$$w_\varphi(\omega) := \langle\omega\,;\varphi\rangle$$

and by using then the $\|\cdot\|_{L^2}$-density of $\mathcal{S}$ in L^2. Note that w is a linear isometry between Hilbert spaces.

Starting from w, one could easily construct a 1D standard Brownian motion $W = (W(t))_{t\in\mathbb{R}}$ on Ω ($\mathbf{P}$-a.s. null for negative times) in such a way that, for any $f \in L^2$,

$$w_f = \int_\mathbb{R} f(t)\,\mathrm{d}W(t)$$

which is a Wiener-Itô integral on $\mathbb{R}$. Let's settle $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ as the filtration generated by W.

For $n \in \mathbb{N}$, let $H_n \in \mathbb{R}[x;n]$ be the n-th Hermite polynomial (on $\mathbb{R}$), which is the n-th coefficient in $x \in \mathbb{R}$ of the series expansion in powers of $t \in \mathbb{R}$ of the smooth function $(t,x) \mapsto \exp(tx - t^2/2)$, namely

$$H_n(x) = \frac{(-1)^n}{n!}\,e^{x^2/2}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\,e^{-x^2/2}$$

and let $(\varphi_n)_{n\in\mathbb{N}^*}$ be the orthonormal basis of L^2 consisting of the Hermite functions: for any $x \in \mathbb{R}$,

$$\varphi_n(x) = \pi^{-1/4}(n-1)!^{-1/2}e^{-x^2/2}H_{n-1}(\sqrt{2}x).$$

Indicate with $\Lambda = (\Lambda,+)$ the space of the multi-indices, i.e. the sequences $k = (k_n)_{n\in\mathbb{N}^*}$ in $\mathbb{N}$ with $k_n \neq 0$ only for a finite number $\ell(k)$ of them, and by the way set: $|k| = \sum_{n=1}^\infty k_n$, $k! = \prod_{n=1}^\infty k_n!$, $0 := (0)_{n\in\mathbb{N}^*}$ and, for $j \in \mathbb{N}^*$, $\delta_{(j)} := (\delta_{j,n})_{n\in\mathbb{N}^*}$. Define, for every $k = (k_n)_{n\in\mathbb{N}^*} \in \Lambda$,

$$\Phi_k := \sqrt{k!}\,\prod_{n=1}^\infty H_{k_n}(w_{\varphi_n})$$

and, for $j \in \mathbb{N}^*$, $\Phi_{(j)} := \Phi_{\delta_{(j)}} = w_{\varphi_j}$. Then $(\Phi_k)_{k\in\Lambda}$ is an orthonormal basis of L^2 and, for any $t \geq 0$,

$$W(t) = \sum_{j=1}^\infty\biggl(\int_0^t \varphi_j(r)\,\mathrm{d}r\biggr)\Phi_{(j)}.$$

A *Hida test function* on Ω is a random variable $X = \sum_{k\in\Lambda} a_k\Phi_k \in L^2(\mathbf{P})$, with $(a_k)_{k\in\Lambda}$ in $\mathbb{R}$, such that if, for any $k = (k_n)_{n\in\mathbb{N}^*} \in \Lambda$, $(k_{(1)}, \dots, k_{(\ell(k))})$ denotes the ordered $\ell(k)$-tuple formed by the non-zero components of k then, for every $p \in \mathbb{R}$,

$$\|X\|^2_{(p)} \coloneqq \sum_{k\in\Lambda} k!\,a_k^2 \prod_{n=1}^{\ell(k)} (2n)^{pk_{(n)}} < \infty.$$

The space of the Hida test functions on Ω is the *Hida test function space* $(\mathcal{S}) \subset L^2(\mathbf{P})$ and is equipped by the projective topology induced by the family of norms $\|\cdot\|_{(p)}$. For instance, for any $t \in \mathbb{R}$, $W(t) \in (\mathcal{S})$.

A *Hida distribution* on Ω is a formal series $Y = \sum_{k\in\Lambda} b_k\Phi_k$, with $(b_k)_{k\in\Lambda}$ in $\mathbb{R}$ and where $\mathbf{E}[Y] \coloneqq b_0$ is named the *generalized expectation* of Y, such that there exists $q \in \mathbb{R}$ for which

$$\|Y\|^2_{(-q)} \coloneqq \sum_{k\in\Lambda} k!\,b_k^2 \prod_{n=1}^{\ell(k)} (2n)^{-qk_{(n)}} < \infty.$$

The space of the Hida distributions on Ω is the *Hida distribution space* and results the dual space $(\mathcal{S})' \supset L^2(\mathbf{P})$ of $(\mathcal{S})$ with $\langle Y\,;X\rangle = \sum_{k\in\Lambda} k!\,a_k b_k$. Here the ω-pointwise product does not make sense.

The *singular (t-pointwise) white noise* on Ω is given by, for any $t \in \mathbb{R}$, the distributional derivative $\dot{W}(t) \in (\mathcal{S})'$ taken in $(\mathcal{S})'$ of $W(t)$ and thus, equivalently,

$$\dot{W}(t) = \sum_{j=1}^{\infty} \varphi_j(t)\,\Phi_{(j)}.$$

Well, for every $X = \sum_{k\in\Lambda} a_k\Phi_k$ and $Y = \sum_{k\in\Lambda} b_k\Phi_k$ in $(\mathcal{S})'$, the *Wick product* $X \diamond Y$ of X and Y is the Hida distribution on Ω defined as

$$X \diamond Y \doteq \sum_{k\in\Lambda} \left(\sum_{\alpha+\beta=k} a_\alpha b_\beta \right) \Phi_k.$$

The Wick algebra obeys the rules of an ordinary algebra, even together with the operation of sum, while caution is required about combining it with the ordinary product: in general, for arbitrary $X, Y, Z \in (\mathcal{S})'$, $X \cdot (Y \diamond Z) \neq (X \cdot Y) \diamond Z$ (whenever those pointwise products do make sense). Note also that in general, for arbitrary $X, Y \in L^2(\mathbf{P})$, $X \diamond Y \notin L^2(\mathbf{P})$ but, if $X, Y \in (\mathcal{S})$, then $X \diamond Y \in (\mathcal{S})$ as well.

A map $Y\colon \mathbb{R}_t \to (\mathcal{S})'$ is said $(\mathcal{S})'$*-integrable* if, for every $X \in (\mathcal{S})$, $\langle Y(\,\cdot\,)\,;X\rangle \in L^1 \coloneqq L^1(\mathbb{R};\mathbb{R})$ and, in such a case, there exists an unique Hida distribution on Ω, the $(\mathcal{S})'$*-integral* $\int_{\mathbb{R}} Y(t)\,\mathrm{d}t$ of Y, with

$$\left\langle \int_{\mathbb{R}} Y(t)\,\mathrm{d}t\,;X \right\rangle = \int_{\mathbb{R}} \langle Y(t)\,;X\rangle\,\mathrm{d}t.$$

We remark that, if $Y(\,\cdot\,)$ is $(\mathcal{S})'$-integrable then, for any $T \in \mathbb{R}_+^*$, $Y(\,\cdot\,)\mathbb{1}_{[0,T]}(\,\cdot\,)$ remains $(\mathcal{S})'$-integrable and we write $\int_0^T Y(t)\,\mathrm{d}t \coloneqq \int_{\mathbb{R}} (Y(t)\mathbb{1}_{[0,T]}(t))\,\mathrm{d}t$.

Theorem. *Fix $T \in \mathbb{R}_+^*$ and take $u = (u(t))_{t=0}^T \in \operatorname{Dom}\delta$ as a process on Ω. Then, for any $t \in [0,T]$, $u(t) \diamond \dot{W}(t)$ is $(\mathcal{S})'$-integrable and its $(\mathcal{S})'$-integral coincides with the Skorohod integral of u (w.r.t. W):*

$$\int_0^T u(t) \diamond \dot{W}(t)\,\mathrm{d}t = \int_0^T u(t)\,\delta W(t) \in L^2(\mathbf{P}).$$

For any map $Y\colon \mathbb{R}_t \to (\mathcal{S})'$ such for which $Y(\,\cdot\,) \diamond \dot{W}(\,\cdot\,)$ is $(\mathcal{S})'$-integrable, $\int_{\mathbb{R}} Y(t) \diamond \dot{W}(t)\,\mathrm{d}t$ is the *generalized Skorohod integral* of Y.

An entire construction like all that could be made in a very more general way and furthermore, anyhow, it could be genuinely connected to the Wiener-Itô chaos expansion.

Rough volatility

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space such that there exists a 2D standard Brownian motion $(W, \overline{W}) = ((W(t), \overline{W}(t)))_{t \in \mathbb{R}}$ on it ($\mathbf{P}$-a.s. null for negative times) and fix $T \in \mathbb{R}_+^*$.

On the one hand put $\rho \in [-1, 1] \setminus \{0\}$, $\overline{\rho} := \sqrt{1-\rho^2}$ and produce the 1D standard Brownian motion $B = (B(t))_{t \in \mathbb{R}}$ on Ω defining its trajectories, for any $t \in \mathbb{R}$, as $B(t) := \rho W(t) + \overline{\rho}\,\overline{W}(t)$ in such a way that B and W have constant correlation $\rho \neq 0$. Assign to Ω the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}}$ generated by B.

On the other hand choose a Hurst index $H \in \,]0, 1/2[$, take the Volterra kernel K on $\mathbb{R}_+^2$ given by, for any $s, t \geq 0$, $\mathrm{K}(s,t) := \mathrm{K}(t-s)$ where, for any $r \in \mathbb{R}$,

$$\mathrm{K}(r) := \sqrt{2H}\,|r|^{H-1/2}\mathbb{1}_{\mathbb{R}_+}(r)$$

and consider the 1D fractional Riemann-Liouville Brownian motion $\widehat{W} = (\widehat{W}(t))_{t \in \mathbb{R}}$ on Ω of index H and Volterra dynamics based on W having trajectories, for any $t \in \mathbb{R}$,

$$\widehat{W}(t) := \int_0^t \mathrm{K}(t-s)\,\mathrm{d}W(s).$$

We remark that $\widehat{W}$ is a continuous Gaussian process with negatively correlated increments and locally H^--Hölder continuous trajectories, i.e. of any order strictly smaller than H and thus rougher than Brownian paths, which is a local martingale independent of $\overline{W}$ that admits quadratic variation (by virtue of the classical Burkholder-Davis-Gundy inequality).

Fixed $f \in C^1(\mathbb{R}; \mathbb{R}_+)$, a *(simple) rough volatility* process $\sigma = (\sigma(t))_{t \geq 0}$ on Ω is explicitly given by

$$\sigma(t) := f\big(\widehat{W}(t)\big),\ t \geq 0$$

meaning it indeed as the volatility process corresponding to the stochastic volatility Itô model

$$\begin{cases} \mathrm{d}S(t)/S(t) = \sigma(t)\mathrm{d}B(t),\ t \geq 0, \\ S(0) \neq 0\ [\mathbf{P}]. \end{cases}$$

We're in the presence of a singular SDE due to the roughness of σ in the sense that, as σ is not even a semi-martingale, in particular it admits no Stratonovich form, closely related to which the absence of Markovianity of the model – although $S = (S(t))_{t \geq 0}$ remains a local martingale – and the lack of a Wong-Zakai type approximation theory for that (and consequently the loss of hope in a successful use of the well-known tools and methods for SDEs).

We submit hereunder a regularity structure for σ which would be the basis for solving the above issues basically providing an approximation theory for stochastic integrals of type

$$\int f\big(\widehat{W}\big)\,\mathrm{d}W.$$

Our task is to build an analysis à la Hairer based on renormalized enhanced noise, incorporating and keeping track of the relevant things we've in mind, so dealing with a mix of even hard algebraic-analytical conditions and, as usual in this area, with the problem of discovering the "right" approximation of the noise and therefore the renormalized approximating models.

Well, employed $\kappa \in \,]0, H[$ and $M = M(H, \kappa) := \max\{m \in \mathbb{N} \mid m(H-\kappa) - 1/2 - \kappa \leq 0\} \in \mathbb{N}^*$, define, always depending on M, the index set

$$A := \{-1/2-\kappa, (H-\kappa) - 1/2 - \kappa, \ldots, M(H-\kappa) - 1/2 - \kappa, 0, H-\kappa, \ldots, M(H-\kappa)\}$$

and the symbols set

$$S := \big\{\Xi, \Xi\mathcal{I}(\Xi), \ldots, \Xi\mathcal{I}(\Xi)^M, \mathbf{1}, \mathcal{I}(\Xi) \ldots, \mathcal{I}(\Xi)^M\big\}$$

attaching to every symbol $\boldsymbol{\tau}$ in S a homogeneity $|\boldsymbol{\tau}|$ in A as follows: $|\mathbf{1}| := 0$, $|\Xi| := -1/2-\kappa$ and, for $m = 1, \ldots, M$, $|\Xi\mathcal{I}(\Xi)^m| := m(H-\kappa) - 1/2 - \kappa$ and $|\mathcal{I}(\Xi)^m| := m(H-\kappa)$. By reading powers of symbols as products with themselves, we see that homogeneities are multiplicative.

Of course Ξ should be interpreted as an abstract representation of the white noise ξ belonging to W, that is $\xi = \dot{W}$ in the distributional sense; while $\mathcal{I}(\,\cdot\,)$ has the intuitive meaning of integration against the Volterra kernel, improperly speaking, and in particular $\mathcal{I}(\Xi)$ would perform $\widehat{W}$.

So let $\mathcal{T} \coloneqq \bigoplus_{\boldsymbol{\tau}\in S} \langle \boldsymbol{\tau} \rangle$ be the model space and $G \coloneqq \{\, \Gamma_h \mid h \in (\mathbb{R}, +) \,\}$ be the structure group where

$$\Gamma_h \mathbf{1} \coloneqq \mathbf{1},\ \Gamma_h \Xi \coloneqq \Xi,\ \Gamma_h \mathcal{I}(\Xi) \coloneqq \mathcal{I}(\Xi) + h\mathbf{1} \text{ and } \Gamma_h \boldsymbol{\tau}\cdot\boldsymbol{\tau'} \coloneqq \Gamma_h \boldsymbol{\tau}\cdot\Gamma_h\boldsymbol{\tau'} \text{ for } \boldsymbol{\tau},\boldsymbol{\tau'} \in S \text{ with } \boldsymbol{\tau}\cdot\boldsymbol{\tau'} \in S$$

then extending to $\mathcal{T}$ by linearity and getting that the triple $\mathscr{T} = (A, \mathcal{T}, G)$ is a regularity structure (also thanks to the basic binomial theorem which will serve throughout the continuation).

With the aim of building an appropriate limiting Itô model $\mathrm{M} = (\Pi, \Gamma)$ for $\mathscr{T}$ (on $\mathbb{R}$) based on ξ, for $m = 1, \dots, M$ we give a meaning to the terms $\Xi\mathcal{I}(\Xi)^m$ by defining an iterated Itô integral $\mathbb{W}^m$ of second order so that it constitutes somehow a kind of its primitive (the context, for the moment, is too irregular to simply propose a product): for $s, t \in \mathbb{R}$ with $s \le t$,

$$\mathbb{W}^m(s,t) \coloneqq \int_s^t \Big(\widehat{W}(r) - \widehat{W}(s)\Big)^m \mathrm{d}W(r).$$

By osserving that this process satisfies $\mathbf{P}$-a.s. the pseudo Chen's relation, for $s, u, t \in \mathbb{R}$ with $s \le u \le t$,

$$\mathbb{W}^m(s,t) = \mathbb{W}^m(s,u) + \sum_{l=0}^{m} \binom{m}{l} \Big(\widehat{W}(u) - \widehat{W}(s)\Big)^l \mathbb{W}^{m-l}(u,t)$$

we extend $\mathbb{W}^m$ to $\mathbb{R}^2$ just by imposing that useful relation for every $s, u, t \in \mathbb{R}$: for $s, t \in \mathbb{R}$ with $t < s$,

$$\mathbb{W}^m(s,t) \coloneqq -\sum_{l=0}^{m} \binom{m}{l} \Big(\widehat{W}(t) - \widehat{W}(s)\Big)^l \mathbb{W}^{m-l}(t,s).$$

Lemma. *For $m = 1, \dots, M$ there exists a version of $\mathbb{W}^m$, for which we keep its symbol, and there exists $p \in [1, \infty[$ such that, for any $K \in \mathcal{K}$, there exists a random positive t-constant $C_K \in L^p(\mathbf{P})$ such that, for any $s, t \in K$ (and $\mathbf{P}$-a.s.),*

$$|\mathbb{W}^m(s,t)| \le C_K |s-t|^{m(H-\kappa)+1/2-\kappa}.$$

For $s, t \in \mathbb{R}$ and $m = 1, \dots, M$ (and $\mathbf{P}$-a.s.), define

$$\begin{cases} \Gamma_{t,s}\mathbf{1} \coloneqq \mathbf{1} \\ \Gamma_{t,s}\Xi \coloneqq \Xi \\ \Gamma_{t,s}\mathcal{I}(\Xi) \coloneqq \mathcal{I}(\Xi) + \big(\widehat{W}(t) - \widehat{W}(s)\big)\mathbf{1} \\ \Gamma_{t,s}\boldsymbol{\tau}\cdot\boldsymbol{\tau'} \coloneqq \Gamma_{t,s}\boldsymbol{\tau}\cdot\Gamma_{t,s}\boldsymbol{\tau'} \text{ when } \boldsymbol{\tau}\cdot\boldsymbol{\tau'} \in S \end{cases} \quad \text{and} \quad \begin{cases} \Pi_s\mathbf{1} \coloneqq 1 \\ \Pi_s\Xi \coloneqq \dot{W} \\ \Pi_s\mathcal{I}(\Xi)^m \coloneqq \big(\widehat{W}(\,\cdot\,) - \widehat{W}(s)\big)^m \\ \Pi_s\Xi\mathcal{I}(\Xi)^m \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{W}^m(s,\,\cdot\,) \end{cases}$$

then extending to $\mathcal{T}$ by linearity.

Proposition. *The pair $\mathrm{M} = (\Pi, \Gamma)$, where $\Gamma = (\Gamma_{t,s})_{t,s\in\mathbb{R}}$ and $\Pi = (\Pi_s)_{s\in\mathbb{R}}$, is ($\mathbf{P}$-a.s.) a model for $\mathscr{T}$.*

The next goal is to find, for $\varepsilon \downarrow 0$, a reasonable approximating model $\mathrm{M}^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon)$ for $\mathscr{T}$ of M, to be renormalized, relying on a smart approximation $\dot{W}^\varepsilon$ of ξ (as distributions). About this, let's consider a function $\delta^\varepsilon\colon \mathbb{R}^2_{(x,y)} \to \mathbb{R}$, which should be understood as an approximation of the Dirac delta and which could be easily built from a mollifier as well as a wavelet basis, with the following properties.

⋆ The function δ^ε is measurable, symmetric and bounded with $\|\delta^\varepsilon\|_\infty = \mathcal{O}(\varepsilon^{-1})$.

⋆ There exists $\beta \in\,]1/2+\kappa, \infty[$ such that, for any $x \in \mathbb{R}$, the function $\delta^\varepsilon(x,\,\cdot\,)\colon \mathbb{R}_y \to \mathbb{R}$ belongs to the *Besov space* $\mathcal{B}^\beta_{1,\infty}(\mathbb{R})$ and $x \mapsto \delta^\varepsilon(x,\,\cdot\,)$ is measurable and bounded as a map from $\mathbb{R}_x$ into $\mathcal{B}^\beta_{1,\infty}(\mathbb{R})$.

⋆ There exists $c \in\,]0, \infty[$ such that, for any $x \in \mathbb{R}$, $\operatorname{supp} \delta^\varepsilon(x,\,\cdot\,) \subset B_{c\varepsilon}(x)$ with $\int_{\mathbb{R}} \delta^\varepsilon(x,y)\,\mathrm{d}y = 1$.

Indeed $\dot W$ turns out to be locally contained in $\mathcal{B}^{-1/2-\kappa}_{\infty,\infty}(\mathbb{R}) \subset \big(\mathcal{B}^{\beta}_{1,\infty}(\mathbb{R})\big)'$ and therefore we define the approximation $\dot W^\varepsilon \coloneqq (\dot W^\varepsilon(t))_{t\in\mathbb{R}}$ of ξ, a Gaussian pathwise measurable locally bounded process, as

$$\dot W^\varepsilon(t) \coloneqq \Big\langle \dot W\,;\delta^\varepsilon(t,\boldsymbol{\cdot})\Big\rangle \mathbb{1}_{\mathbb{R}_+}(t).$$

For 1D stochastic process $u=(u(t))_{t=0}^T$ on Ω and $t\in\mathbb{R}_+$, we write

$$\int_0^t u(r)\,\mathrm{d}W^\varepsilon(r) \coloneqq \int_0^t u(r)\dot W^\varepsilon(r)\,\mathrm{d}r$$

while, if u takes values in some non-homogeneous *Wiener chaos* induced by $\dot W$, we write

$$\int_0^t u(r)\diamond \mathrm{d}W^\varepsilon(r) \coloneqq \int_0^t u(r)\diamond \dot W^\varepsilon(r)\,\mathrm{d}r.$$

In particular, we consider the approximation $\widehat{W}^\varepsilon \coloneqq (\widehat{W}^\varepsilon(t))_{t\in\mathbb{R}}$ of $\widehat{W}^\varepsilon$ as

$$\widehat{W}^\varepsilon(t) \coloneqq \mathrm{K} * \dot W^\varepsilon = \int_0^t \mathrm{K}(t-r)\,\mathrm{d}W^\varepsilon(r).$$

Lemma. *For $\varepsilon\downarrow 0$, there exist $p\in[1,\infty[$ and random positive t-constants $C_{\varepsilon,T}, C_T\in L^p(\mathbf{P})$, where are uniformly bounded, such that, for any $s,t\in[0,T]$, $\kappa'\in\,]0,H[$ and $\delta\in\,]0,1[$ (and $\mathbf{P}$-a.s.),*

$$\Big|\widehat{W}^\varepsilon(t)-\widehat{W}^\varepsilon(s)\Big| \le C_{\varepsilon,T}|t-s|^{H-\kappa'} \quad \textit{and} \quad \Big|\widehat{W}^\varepsilon(t)-\widehat{W}^\varepsilon(s)-\Big(\widehat{W}(t)-\widehat{W}(s)\Big)\Big| \le C_T|t-s|^{H-\kappa'}\varepsilon^{\delta\kappa'}.$$

For $\varepsilon\downarrow 0$, $s,t\in\mathbb{R}$ and $m=1,\dots,M$ (and $\mathbf{P}$-a.s.), define the approximating model for $\mathscr{T}$ of M by

$$\begin{cases}\Gamma^\varepsilon_{t,s}\mathbf{1} \coloneqq \mathbf{1}\\ \Gamma^\varepsilon_{t,s}\Xi \coloneqq \Xi\\ \Gamma^\varepsilon_{t,s}\mathcal{I}(\Xi) \coloneqq \mathcal{I}(\Xi) + \big(\widehat{W}^\varepsilon(t)-\widehat{W}^\varepsilon(s)\big)\mathbf{1}\\ \Gamma^\varepsilon_{t,s}\boldsymbol{\tau}\cdot\boldsymbol{\tau'} \coloneqq \Gamma^\varepsilon_{t,s}\boldsymbol{\tau}\cdot\Gamma^\varepsilon_{t,s}\boldsymbol{\tau'} \text{ when } \boldsymbol{\tau}\cdot\boldsymbol{\tau'}\in S\end{cases} \quad\text{and}\quad \begin{cases}\Pi^\varepsilon_s\mathbf{1} \coloneqq 1\\ \Pi^\varepsilon_s\Xi \coloneqq \dot W^\varepsilon\\ \Pi^\varepsilon_s\mathcal{I}(\Xi)^m \coloneqq \big(\widehat{W}^\varepsilon(\boldsymbol{\cdot})-\widehat{W}^\varepsilon(s)\big)^m\\ \Pi^\varepsilon_s\Xi\mathcal{I}(\Xi)^m \coloneqq \dot W^\varepsilon(\boldsymbol{\cdot})\big(\widehat{W}^\varepsilon(\boldsymbol{\cdot})-\widehat{W}^\varepsilon(s)\big)^m\end{cases}$$

(extending then to $\mathcal{T}$ by linearity).

Proposition. *The pair $\mathrm{M}^\varepsilon=(\Pi^\varepsilon,\Gamma^\varepsilon)$, where $\Gamma^\varepsilon=(\Gamma^\varepsilon_{t,s})_{t,s\in\mathbb{R}}$ and $\Pi^\varepsilon=(\Pi^\varepsilon_s)_{s\in\mathbb{R}}$, is a model for $\mathscr{T}$.*

In order to understand the diverging quantities $\mathscr{C}^\varepsilon$ to be subtracted from Π^ε to renormalize it in the sense of Hairer, since there cannot be any hope of a convergence like $\int f\big(\widehat{W}^\varepsilon\big)\mathrm{d}W^\varepsilon$ to $\int f\big(\widehat{W}\big)\mathrm{d}W$, what's below is enlightening. Before, given 1D Gaussians U_1,V,U_2 on Ω and $l,h\in\mathbb{N}^*$ with $l\le h$,

$$U_1^l\cdot(V\diamond U_2^{h-l}) = V\diamond(U_1^lU_2^{h-l}) + l\,\mathbf{E}[VU_1]U_1^{l-1}U_2^{h-l}.$$

Lemma. *For any $\varphi\in\mathcal{D}$, $s\in\mathbb{R}$, $m=1,\dots,M$ and $\varepsilon\downarrow 0$, the two following identities hold.*

1. $\Pi_s\Xi\mathcal{I}(\Xi)^m(\varphi) = \displaystyle\int_0^\infty \varphi(t)\big(\widehat{W}(t)-\widehat{W}(s)\big)^m\delta W(t) - m\int_0^s \varphi(s)\mathrm{K}(s-r)\big(\widehat{W}(r)-\widehat{W}(s)\big)^{m-1}\mathrm{d}r.$

2. *Defined $\mathscr{K}^\varepsilon(s,\boldsymbol{\cdot}) \coloneqq \mathbf{E}\big[\widehat{W}^\varepsilon(s)\dot W^\varepsilon(\boldsymbol{\cdot})\big] = \mathbb{1}_{\mathbb{R}^2_+}(s,\boldsymbol{\cdot})\displaystyle\int_0^\infty\int_0^\infty \delta^\varepsilon(\boldsymbol{\cdot},x)\delta^\varepsilon(x,y)\mathrm{K}(s-y)\,\mathrm{d}x\mathrm{d}y$, then*

$$\Pi^\varepsilon_s\Xi\mathcal{I}(\Xi)^m(\varphi) = \int_0^\infty \varphi(t)\big(\widehat{W}^\varepsilon(t)-\widehat{W}^\varepsilon(s)\big)^m\diamond\mathrm{d}W^\varepsilon(t) - m\int_0^\infty \varphi(t)\left[\mathscr{K}^\varepsilon(s,t)-\mathscr{K}^\varepsilon(t,t)\right]\big(\widehat{W}^\varepsilon(t)-\widehat{W}^\varepsilon(s)\big)^{m-1}\mathrm{d}t$$

Furthermore, assuming $s\le T$, there exists $C_T\in\mathbb{R}^*_+$ such that, for any $t\in[0,T]$,

$$|\mathscr{K}^\varepsilon(s,t)| \le C_T\,\varepsilon^{H-1/2}.$$

As a corollary, if we interpret $\mathscr{K}^\varepsilon$ as an approximation of the kernel K, then we see that

$$\mathscr{C}^\varepsilon(t) \coloneqq \mathscr{K}^\varepsilon(t,t), \quad t \geq 0$$

would correspond to something diverging like "$0^{H-1/2} = \infty$" in the limit $\varepsilon \downarrow 0$.

Theorem. *For $\varepsilon \downarrow 0$, $s \in \mathbb{R}$ and $m = 1, \ldots, M$ (and $\mathbf{P}$-a.s.), define*

$$\widehat{\Pi}_s^\varepsilon \Xi \mathcal{I}(\Xi)^m \coloneqq \Pi_s^\varepsilon \Xi \mathcal{I}(\Xi)^m - m \mathscr{C}^\varepsilon(\,\bullet\,) \Pi_s^\varepsilon \mathcal{I}(\Xi)^{m-1}$$

leaving $\widehat{\Pi}_s^\varepsilon \boldsymbol{\tau} \coloneqq \Pi_s^\varepsilon \boldsymbol{\tau}$ on the remaining symbols $\boldsymbol{\tau} \in S$. Then the pair $\widehat{\mathrm{M}}^\varepsilon = (\widehat{\Pi}^\varepsilon, \Gamma^\varepsilon)$, where $\widehat{\Pi}^\varepsilon = (\widehat{\Pi}_s^\varepsilon)_{s=0}^T$, is a model for $\mathscr{T}$ and there exists $C_T \in \mathbb{R}_+^*$ such that, for any $p \in [1, \infty[$ and $\delta \in]0,1[$,

$$\Big\| \big|\big|\big| \widehat{\mathrm{M}}^\varepsilon ; \mathrm{M} \big|\big|\big|_T \Big\|_{L^p(\mathbf{P})} \leq C_T\, \varepsilon^{\delta\kappa}$$

where of course, for any model $\widetilde{\mathrm{M}} = (\widetilde{\Pi}, \widetilde{\Gamma})$ for $\mathscr{T}$ on $\mathbb{R}$,

$$\begin{aligned} \big|\big|\big| \widetilde{\mathrm{M}} ; \mathrm{M} \big|\big|\big|_T \doteq & \sup \Big\{ \Big| \big(\Pi_s - \widetilde{\Pi}_s\big) \boldsymbol{\tau}(\varphi_s^\lambda) \Big| \lambda^{-|\boldsymbol{\tau}|} \;\Big|\; \varphi \in \mathcal{D}_1, \lambda \in]0,1], s \in [0,T], \boldsymbol{\tau} \in S \Big\} \\ & + \sup \Big\{ \Big\| \Gamma_{t,s} \boldsymbol{\tau} - \widetilde{\Gamma}_{t,s} \boldsymbol{\tau} \Big\|_{\alpha'} |t-s|^{\alpha' - |\boldsymbol{\tau}|} \;\Big|\; t,s \in [0,T], \boldsymbol{\tau} \in S, \alpha' \in A \text{ with } \alpha' < |\boldsymbol{\tau}| \Big\}. \end{aligned}$$

Starting reference

[1] C. Bayer, P. K. Friz, P. Gassiat, J. Martin, B. Stemper. *A regularity structure for rough volatility.* Mathematical Finance (2019), pp. 1–51.