Multiobjective stochastic problems and their connections with set-valued risk measures

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Abstract

In this paper we generalize the study of minimax stochastic programming to the case where the objective function is multiobjective. In doing this, we will consider the set approach. We will provide necessary and sufficient condition of optimality in terms of suitable first order conditions.

Then, we compare the proposed approaches with the minimization of set-valued risk-measures recently introduced by Jouini et al. (2004); Cascos and Molchanov (2007) and Hamel and Heyde (2010). We show that the set-minimization of a certain of set-valued risk measure is, in a sense, equivalent to optimize a set-valued expected value problem with respect to some weighted distribution in the set of admissible distributions. We also introduce and analyze specific optimization problems involving risk functions.

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1 Introduction

Consider a stochastic system with a n-dimensional random vector $\mathbf{X}$ on a measurable space $(\Omega, \mathscr{F})$. If it depends on a decision vector $\mathbf{x} \in \mathbb{R}^k$, we can write:

$$\mathbf{X}(\omega) = \varphi(\mathbf{x}, \omega), \quad \omega \in \Omega.$$

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Here, $\varphi : \mathbb{R}^k \times \Omega \to \mathbb{R}^n$ is assumed to be measurable. We focus on the case where $\mathbf{X}$ represents the uncertain value of some financial positions.

To find the best value of the decision vector $\mathbf{x}$, one approach (see, e.g., Birge and Louveaux (2011) or Shapiro (2003)) is to optimize the expected value of φ with respect to a vector of probability measures $\mathbb{P} := (\mathbb{P}_1, \ldots, \mathbb{P}_n)$ on $(\Omega, \mathcal{F})$, while keeping the decision variable feasible. The problem can be formulated as:

$$\begin{aligned} &\text{minimize} \;\; \mathbf{f}(\mathbf{x}) \\ &\text{s.t. } \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^k \end{aligned} \tag{1}$$

with $\mathbf{f} : \mathbb{R}^k \to \mathbb{R}^n$ defined as

$$\mathbf{f}(\mathbf{x}) := \begin{pmatrix} \mathbb{E}_{\mathbb{P}_1}[\varphi_1(\mathbf{x}, \omega)] \\ \vdots \\ \mathbb{E}_{\mathbb{P}_n}[\varphi_n(\mathbf{x}, \omega)] \end{pmatrix}.$$

Note that (1) is a multiobjective stochastic optimization problem; thus, unlike the scalar case, the exact meaning of "minimize" still needs to be specified. Additionally, in a financial context, optimizing the expected value of a loss distribution provides an optimal decision only on average and often does not sufficiently represent the potential loss due to data variability. Moreover, problem (1) implicitly assumes that the vector of probability distributions $(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ is known, which is a simplification.

In the literature, there are several ways to address these difficulties. In the one-dimensional case (i.e., when $\mathbf{f}$ is real-valued), one approach is to identify a plausible family $\mathcal{Q}$ of probability distributions and replace the original objective function (depending on a single probability measure $\mathbb{P}$) with the worst-case expectation:

$$\sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}\big[\varphi(\mathbf{x}, \omega)\big]$$

(here φ is a scalar function).

With this approach, the optimization problem

$$\begin{aligned} &\text{minimize} \quad \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x}, \omega)] \\ &\text{s.t. } \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^k \end{aligned} \tag{2}$$

is called the robust counterpart of the problem

$$\begin{aligned} &\text{minimize} \;\; \mathbb{E}_{\mathbb{P}}[\varphi(\mathbf{x}, \omega)] \\ &\text{s.t. } \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^k. \end{aligned}$$

This notion can be found in Ben-Tal et al. (2009) and guarantees worst-case performance. Various other concepts of robust solutions have been introduced in the literature, see, e.g., Bertsimas and Sim (2004), Ben-Tal et al. (2009), Fischetti and Monaci (2009), and Garcia and Pena (2018).

The first aim of this paper is to extend the investigation of robust problems to problems of the form (2), i.e., to problems in the more general multiobjective framework, and to illustrate the link between such problems and the minimization of set-valued risk measures as introduced in Cascos and Molchanov (2007); Hamel and Heyde (2010). To proceed, we apply the techniques developed in Ehrgott et al. (2014). We consider the following approach.

Set-robust counterpart of problem (1): minimize the set-valued functional $\mathbf{F}_{\mathcal{Q}}(\mathbf{x})$ defined as

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) := \left\{ \begin{bmatrix} \mathbb{E}_{\mathbb{Q}_1}[\varphi_1(\mathbf{x},\omega)] \\ \vdots \\ \mathbb{E}_{\mathbb{Q}_d}[\varphi_d(\mathbf{x},\omega)] \end{bmatrix} : (\mathbb{Q}_1,\dots,\mathbb{Q}_d) \in \mathcal{Q} \right\} \tag{3}$$
$$\text{s.t. } \mathbf{x} \in \mathcal{X}.$$

Here, the minimization is understood in terms of a suitable order relation between sets, according to which appropriate notions of robust solutions can be given (see Section 2.1 for details). We will characterize robust solutions of problem (3) by means of global minima of a suitable nonlinear scalar function of $\mathbf{x}$. In addition, we will provide necessary and sufficient conditions for a pair $(\bar{\mathbf{x}}, \bar{\mathbb{Q}})$ to be optimal in terms of the subdifferential of the associated nonlinear scalarization.

The second part of the paper compares the set-approach with risk minimization in a set-valued framework. Measuring the risk of a random payoff involves computing quantities that summarize the loss distribution of the portfolio over a predetermined horizon. Examples include set-valued extensions of well-known risk measures such as Value-at-Risk and Expected Shortfall (see Jouini et al. (2004), Cascos and Molchanov (2007), and Hamel and Heyde (2010)). Minimizing such measures can be formulated as follows:

Risk minimization problem in the set case. Given $\psi : \mathbb{R}^k \to \mathcal{Y}$ and a set-valued risk measure $\mathbf{R}$ in the spirit of Hamel and Heyde (2010) (see also Cascos and Molchanov (2007) and Jouini et al. (2004)), minimize (with respect to a suitable set-order relation) the set-valued composite risk measure

$$\mathbf{R}\big(\psi(\mathbf{x})\big), \tag{4}$$

over the feasible set $\mathcal{X} \subseteq \mathbb{R}^k$.

We will provide conditions on ψ that ensure $\mathbf{R}(\psi(\cdot))$ satisfies convexity. Additionally, we will show that problems of type (3) correspond to the minimization of a specific class of risk measures. Finally, we will present examples involving a set-valued risk measure to illustrate the effectiveness of the proposed methods.

The approach taken here complements the one in Mastrogiacomo et al. (2025), which focuses on the so-called componentwise robust counterpart. Inspired by Fliege and Werner (2014), the approach developed in Mastrogiacomo et al. (2025) aims to find the set of efficient points of the multiobjective function $\mathbf{f}_{\mathcal{Q}} : \mathbb{R}^k \to \mathbb{R}^n$, given a nonempty set $\mathcal{Q}$ of vectors of probability measures (the set of admissible distributions or reference probabilities). Such function is defined as:

$$\mathbf{f}_{\mathcal{Q}}(\mathbf{x}) := \begin{bmatrix} \sup_{\mathbb{Q}_1 \in \mathcal{Q}_1} \mathbb{E}_{\mathbb{Q}_1}[\varphi_1(\mathbf{x},\omega)] \\ \vdots \\ \sup_{\mathbb{Q}_n \in \mathcal{Q}_n} \mathbb{E}_{\mathbb{Q}_n}[\varphi_n(\mathbf{x},\omega)] \end{bmatrix}, \tag{5}$$

over the feasible set $\mathcal{X} \subseteq \mathbb{R}^k$. Here, $\mathcal{Q}_i$, for $i = 1, \ldots, d$, denotes the projection of the closure $\operatorname{cl}\operatorname{co}\mathcal{Q}$ on the i-th component.

In Mastrogiacomo et al. (2025), the authors show that, under suitable regularity conditions, there exists an n-dimensional vector of probability distributions $\bar{\mathbb{Q}} = (\bar{\mathbb{Q}}_1, \ldots, \bar{\mathbb{Q}}_n)$ on the convex hull of the set of admissible distributions in $\mathcal{Q}$ such that the component-wise robust counterpart problem is equivalent to the multiobjective optimization problem associated with $\bar{\mathbb{Q}}$. The authors also provided necessary and sufficient conditions for a pair $(\bar{\mathbf{x}}, \bar{\mathbb{Q}})$ to be optimal in terms of the subdifferential of a suitable scalar function.

Instead, the robustification approach considered in this paper is inspired by Ehrgott et al. (2014), and consists in treating the robust set-valued version of the multiobjective stochastic optimization problem (1).

The paper is organized as follows: Section 2 addresses robust multiobjective stochastic optimization problems and characterizes solutions using suitable scalarizations and their subdifferentials. Subsection 2.1 provides an overview of the necessary tools from set optimization, while Subsection 2.2 focuses on the robust counterpart approach proposed in Ehrgott et al. (2014). Section 3 deals with set-valued risk measures. Subsection 3.1 recalls the basic definitions of set-valued risk measures, while Subsection 3.2 establishes the link with the set-robust counterpart of (1) and discusses the properties of composite set-valued risk measures. Finally, Section 4 presents illustrative numerical examples from portfolio optimization under set-valued risk measures, providing further insights into the proposed robust counterparts.

2 Robust multiobjective stochastic problems

2.1 Basic notions and definitions

In this subsection we recall some basic notions and notations from set optimization theory used in the paper (in particular in Subsection 2.2). We also recall some basic notions from probability theory required for the development of the second part of the paper.

Multiobjective optimization. Multiobjective optimization deals with the problem of minimizing (in a suitable sense) a function $\mathbf{f} : \mathcal{X} \subseteq \mathbb{R}^k \to \mathbb{R}^n$ subject to some constraints defined by the feasible set $\mathcal{X}$. The exact meaning of

$$\begin{aligned} &\text{minimize} \;\; \mathbf{f}(\mathbf{x}) \\ &\text{s.t. } \mathbf{x} \in \mathcal{X} \end{aligned} \tag{MOP}$$

has to be specified, since there is no standard total order for the image space of $\mathbf{f}(\mathcal{X})$. In this paper, multicriteria problems are defined with respect to the canonical component-wise ordering induced by the cone $\mathbb{R}^n_+$.

Efficient solutions are defined in the following way (see, e.g. Geoffrion (1968)):

Definition 1 *Given the optimization problem* (MOP)*, a feasible solution* $\bar{\mathbf{x}} \in \mathcal{X}$ *is called:*

- *weak efficient if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{f}(\mathbf{x}) \leq_{\mathrm{int}\mathbb{R}^n_+} \mathbf{f}(\bar{\mathbf{x}});$$

- *efficient if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{f}(\mathbf{x}) \leq_{\mathbb{R}^n_+ \setminus \{0\}} \mathbf{f}(\bar{\mathbf{x}});$$

- *strictly efficient if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{f}(\mathbf{x}) \leq_{\mathbb{R}^n_+} \mathbf{f}(\bar{\mathbf{x}}).$$

Clearly, strict efficiency implies efficiency, which in turn leads to weak efficiency.

Remark 2 *We will indicate the relation order* $\leq_{\mathrm{int}\mathbb{R}^n_+}$ *as* $<_{\mathbb{R}^n_+}$*. With this notation, weak solutions can be equivalently characterized as follows: there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that* $\mathbf{f}(\mathbf{x}) <_{\mathbb{R}^n_+} \mathbf{f}(\bar{\mathbf{x}})$*.*

Set-optimization problems. In the last decades, set-valued analysis has been developed and many concepts and properties for set-valued maps had been studied, see, e.g. Corley (1987, 1988); Kuroiwa (1996). In the following we present the main concepts of set-valued maps suitable for set-valued optimization and we focus on u-type minimal solutions. We refer to Kuroiwa (2003) for a more detailed treatment of this subject.

Definition 3 *(See, e.g. Kuroiwa (1998, 1999)) Let* $A, B \subseteq \mathbb{R}^n_+$ *be arbitrarily chosen sets. Then the u-type set relation* $\preceq^u_{\mathbb{R}^n_+}$ *is defined by*

$$A \preceq^u_{\mathbb{R}^n_+} B \qquad \Longleftrightarrow \qquad A \subseteq B - \mathbb{R}^n_+.$$

Consider now a set-valued map $\mathbf{F} : \mathbb{R}^d \to 2^{\mathbb{R}^n}$; then, the set-valued optimization problem with respect to $\preceq^u_{\mathbb{R}^n_+}$ is indicated as

$$\begin{aligned} &\preceq^u_{\mathbb{R}^n_+} \text{ - minimize } \mathbf{F}(\mathbf{x}) \\ &\text{s.t. } \mathbf{x} \in \mathcal{X} \end{aligned} \tag{SP}$$

and u-type minimal solutions are defined in the following way:

Definition 4 *Given a set-valued optimization problem* (SP)*, an element* $\bar{\mathbf{x}} \in \mathcal{X}$ *is called*

- *a minimal solution if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}(\mathbf{x}) \preceq^u_{\mathbb{R}^n_+ \setminus \{0\}} \mathbf{F}(\bar{\mathbf{x}});$$

- *a weak minimal solution if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}(\mathbf{x}) \preceq^u_{\operatorname{int}\mathbb{R}^n_+} \mathbf{F}(\bar{\mathbf{x}});$$

- *a strict minimal solution if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}(\mathbf{x}) \preceq^u_{\mathbb{R}^n_+} \mathbf{F}(\bar{\mathbf{x}}).$$

Remark 5 *We will indicate the relation order* $\prec_{\mathbb{R}^n_+}$ *as* $\operatorname{int}\mathbb{R}^n_+$*. With this notation, weak solutions can be equivalently characterized as follows: there is no* $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{F}(\mathbf{x}) \prec^u_{\mathbb{R}^n_+} \mathbf{F}(\bar{\mathbf{x}})$*.*

2.2 Robust counterpart of stochastic multiobjective optimization problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We denote by $\mathcal{M}_{1,f}(\mathbb{P})$ the set of all (σ-additive) probability measures $\mathbb{Q}$ on $\mathcal{F}$ that are absolutely continuous with respect to $\mathbb{P}$.

$L^\infty_d(\Omega, \mathcal{F}, \mathbb{P})$ (abbreviated as L^∞_d) will indicate the set of essentially bounded d-dimensional random variables. When $d = 1$ we will simply use the notation $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ or the abbreviation L^∞. For $X \in L^\infty$, $\mathbb{E}_{\mathbb{Q}}[X]$ will denote the integral of X with respect to $\mathbb{Q} \in \mathcal{M}_{1,f}(\mathbb{P})$.

We also introduce the convex hull

$$\mathcal{C} := \operatorname{co} \mathcal{Q},$$

and we denote by $\mathcal{C}_i$ each projection of $\mathcal{C}$ onto the i-th component.

We assume the following.

Assumption 6

- $\mathcal{X}$ is a nonempty, closed, and convex subset of $\mathbb{R}^k$.
- There exists a convex neighborhood V of $\mathcal{X}$—that is, a convex set containing an open neighborhood of every point in $\mathcal{X}$—such that $\varphi = (\varphi_1, \ldots, \varphi_n)^\top$ is an $\mathbb{R}^n$-valued measurable mapping defined on $V \times \Omega$, and the following three properties are satisfied.
  - For all $\mathbf{x} \in V$ and $i = 1, \ldots, n$, the function $\varphi_i(\mathbf{x}, \cdot)$ belong to $L^\infty(\mathbb{P})$.
  - For every $\omega \in \Omega$ and $i = 1, \ldots, n$, the function $\varphi_i(\cdot, \omega)$ is convex on V.

– For all $i = 1, \ldots, n$, the sup-function

$$\sup_{\mathbb{Q}_i \in \mathcal{C}_i} \mathbb{E}_{\mathbb{Q}_i}[\varphi_i(\,\cdot\,, \omega)]$$

is finite valued on V.

(A4) $\mathcal{Q}$ is a nonempty and weak*-closed subset of $\mathrm{ba}_n(\mathbb{P})$ ($\mathrm{ba}_n(\mathbb{P})$ is the space of finitely additive bounded set-valued functions on $(\Omega, \mathcal{F})$ whose components are absolutely continuous with respect to $\mathbb{P}$).

For all $i = 1, \ldots, n$, let $g_i \colon V \times \mathcal{C}_i \to \mathbb{R}$ be defined as

$$g_i(\mathbf{x}, \mathbb{Q}_i) := \mathbb{E}_{\mathbb{Q}_i}[\varphi_i(\mathbf{x}, \omega)], \quad (\mathbf{x}, \mathbb{Q}_i) \in V \times \mathcal{C}_i, \tag{6}$$

and, thus, let

$$g(\mathbf{x}, \mathbb{Q}) := \big(g_1(\mathbf{x}, \mathbb{Q}_1), \ldots, g_n(\mathbf{x}, \mathbb{Q}_n)\big)^\top, \quad \mathbf{x} \in V,\ \mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_n) \in \mathcal{C}. \tag{7}$$

In the following, we consider the robust counterpart of the family of multi-objective problems

$$\min_{\mathbf{x} \in \mathcal{X}} g(\mathbf{x}, \mathbb{Q}), \tag{$\mathrm{MOP}_{\mathbb{Q}}$}$$

for $\mathbb{Q} \in \mathcal{C}$, in the spirit of Ehrgott et al. (2014). Namely, the objective-wise worst-case of problem ($\mathrm{MOP}_{\mathbb{Q}}$) is defined as

$$\min_{x \in \mathcal{X}} \begin{bmatrix} \sup_{\mathbb{Q}_1 \in \mathcal{C}_1} \mathbb{E}_{\mathbb{Q}_1}[\varphi_1(\mathbf{x}, \omega)] \\ \vdots \\ \sup_{\mathbb{Q}_n \in \mathcal{C}_n} \mathbb{E}_{\mathbb{Q}_n}[\varphi_n(\mathbf{x}, \omega)] \end{bmatrix}, \tag{$\mathrm{OWC}_{\mathcal{C}}$}$$

i.e., each component of the objective function is replaced by its robust counterpart.

On the other hand, we then consider the set-optimization problem

$$\begin{aligned} &\preceq^u_{\mathbb{R}^n_+} \text{- minimize } \mathbf{F}_{\mathcal{Q}}(\mathbf{x}) := \big\{\mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x}, \omega)] : \ \mathbb{Q} \in \mathcal{Q}\big\}, \\ &\text{s.t. } \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{$\mathrm{RSO}_{\mathcal{Q}}$}$$

Clearly, the function $\mathbf{F}_{\mathcal{Q}}$ is set-valued, i.e., $\mathbf{F}_{\mathcal{Q}} : \mathbb{R}^n \to 2^{\mathbb{R}^d}$.

We quote from (Ehrgott et al., 2014, Definition 3.1) (see also Ide et al. (2014)) the definition of robust (weakly/·/strictly) efficient solution for the robust optimization problem ($\mathrm{RSO}_{\mathcal{Q}}$):

Definition 7 *(See e.g. (Ehrgott et al., 2014, Definition 3.1)) Let* $\mathbf{F} : \mathbb{R}^n \times \mathcal{Q} \to \mathbb{R}^d$*. Given the uncertain multiobjective problem*

$$\min_{\mathbf{x} \in \mathcal{X}} \mathbf{F}(\mathbf{x}, \mathbb{Q}),$$

and

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) := \{\mathbf{F}(\mathbf{x}, \mathbb{Q}) : \ \mathbb{Q} \in \mathcal{Q}\},$$

we call a feasibile solution $\bar{\mathbf{x}} \in \mathcal{X}$*:*

- *robust weakly efficient if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \subseteq \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}) - \operatorname{int} \mathbb{R}^d_+;$$

- *robust efficient if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \subseteq \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}) - \mathbb{R}^d_+ \setminus \{0\};$$

- *robust strictly efficient if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \subseteq \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}) - \mathbb{R}^d_+.$$

Remark 8 *We can reformulate the above definitions in terms of the u-type set-order relations with respect to the cones* $\operatorname{int}\mathbb{R}^d_+$, $\mathbb{R}^d_+ \setminus \{0\}$, *and* $\mathbb{R}^d_+$. *In particular,* $\bar{\mathbf{x}}$ *is:*

- *robust weakly efficient if and only if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \preceq^u_{\operatorname{int}\mathbb{R}^d_+} \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}),$$

or, equivalently,

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \prec^u_{\mathbb{R}^d_+} \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}});$$

- *robust efficient if and only if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \preceq^u_{\mathbb{R}^d_+ \setminus \{0\}} \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}});$$

- *robust strictly efficient if and only if there is no* $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ *such that*

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}) \preceq^u_{\mathbb{R}^d_+} \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}).$$

In this section we show that, under mild regularity conditions, the set-valued robust problem counterpart of the uncertain multiobjective problem ($\mathrm{RSO}_{\mathcal{Q}}$) generates a set of probability distributions on the set of admissible distributions with the robust problem being in some sense equivalent to a scalar minimax problem involving the expected value problem with respect to some weighted distributions.

We begin our investigation by analyzing how weak efficient solutions of ($\mathrm{OWC}_{\mathcal{C}}$) are related to robust weak solutions of ($\mathrm{RSO}_{\mathcal{Q}}$).

Proposition 9 *Let* $\bar{\mathbf{x}} \in \mathcal{X}$ *be a weak efficient solution for problem* ($\mathrm{OWC}_{\mathcal{C}}$). *Then,* $\bar{\mathbf{x}}$ *is a robust weak efficient solution for problem* ($\mathrm{RSO}_{\mathcal{Q}}$).

Proof. Assume $\bar{\mathbf{x}}$ is not a robust weak efficient solution of ($\mathrm{RSO}_{\mathcal{Q}}$). Then there exists $\mathbf{x}_0 \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ such that

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}_0) \prec^u_{\mathbb{R}^d_+} \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}), \quad \text{i.e.,}$$

$$\mathbf{F}_{\mathcal{Q}}(\mathbf{x}_0) \subseteq \mathbf{F}_{\mathcal{Q}}(\bar{\mathbf{x}}) - \mathrm{int}\mathbb{R}^d_+.$$

Hence, for any $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d) \in \mathcal{Q}$ and $i = 1, \ldots, d$, there exist $\bar{\mathbb{Q}}^{(i)} \in \mathcal{Q}$ and $\delta_i > 0$ such that

$$g_i(\mathbf{x}_0, \mathbb{Q}_i) = g_i(\bar{\mathbf{x}}, \bar{\mathbb{Q}}_i^{(i)}) - \delta_i$$

(see (6) and (RSO$_{\mathcal{Q}}$)). It clearly follows that, for all $i = 1, \ldots, d$,

$$\sup_{\mathbb{Q}_i \in \mathcal{Q}_i} g_i(\mathbf{x}_0, \mathbb{Q}_i) \leq \sup_{\mathbb{Q}_i \in \mathcal{Q}_i} g_i(\bar{\mathbf{x}}, \mathbb{Q}_i),$$

which contradicts that $\bar{\mathbf{x}}$ is a weak efficient solution of (OWC$_C$). ■

We proceed now with a first simple characterization of set inclusions. This result is a key lemma for the characterization of robust weak and strictly efficient solutions by means of a nonlinear scalarization approach.

Lemma 10 (See (Jahn, 2015, Lemma 2.1 and Remark 2.1)) *Let A and B be nonempty subsets of $\mathbb{R}^d$ partially ordered with respect to $\mathbb{R}^d_+$. If $A - \mathbb{R}^d_+$ is closed and convex, then*

$$A \preceq^u_{\mathbb{R}^d_+} B \qquad \Longleftrightarrow \qquad \text{for all } v \in \mathbb{R}^d_+ \setminus \{0\}\text{: } \sup_{a \in A} \langle v, a \rangle \leq \sup_{b \in B} \langle v, b \rangle,$$

$$A \prec^u_{\mathbb{R}^d_+} B \qquad \Longleftrightarrow \qquad \text{for all } v \in \mathbb{R}^d_+ \setminus \{0\}\text{: } \sup_{a \in A} \langle v, a \rangle < \sup_{b \in B} \langle v, b \rangle.$$

Remark 11 We emphasize that, in the previous lemma, we can reduce the set of $v \in \mathbb{R}^d_+ \setminus \{0\}$ with a base of $\mathbb{R}^d_+ \setminus \{0\}$ (see, e.g., Kuroiwa and Lee (2012) for the precise definition of base of a cone). In particular, since any $v \in \mathbb{R}^d_+ \setminus \{0\}$ can be written as $v = \alpha w$, where $\alpha > 0$ and w belongs to the unit sphere $\mathbb{S}^1_d := \left\{ z \in \mathbb{R}^d_+ \setminus \{0\} : \|z\|_d = 1 \right\}$ the inequality

$$\sup_{a \in A} \langle v, a \rangle \leq \sup_{b \in B} \langle v, b \rangle, \qquad v \in \mathbb{R}^d_+ \setminus \{0\}$$

is equivalent to

$$\sup_{a \in A} \langle w, a \rangle \leq \sup_{b \in B} \langle w, b \rangle, \qquad w = v/\alpha \in \mathbb{S}^1_d, \text{ for some } \alpha > 0.$$

In the following, we provide a characterization of strict and weak efficient solutions for (RSO$_{\mathcal{Q}}$). For this purpose, for any $\bar{\mathbb{Q}} = (\bar{\mathbb{Q}}^{(1)}, \ldots, \bar{\mathbb{Q}}^{(h)}) \in C^h$, let

$$\begin{aligned} \mathbf{F}_{\bar{\mathbb{Q}}} : \ V &\to \qquad \mathbb{R}^{h \times d} \\ \mathbf{x} &\mapsto \begin{bmatrix} g(\mathbf{x}, \bar{\mathbb{Q}}^{(1)}) \\ \vdots \\ g(\mathbf{x}, \bar{\mathbb{Q}}^{(h)}) \end{bmatrix} \end{aligned}$$

(see (7)). We work also under the following additional assumption.

Assumption 12 *Let* $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$. *Set*

$$f_{\lambda,\mathcal{Q}}(\mathbf{x}) := \sum_{i=1}^{n} \lambda_i \sup_{\mathbb{Q}_i \in \mathcal{C}_i} \mathbb{E}_{\mathbb{Q}_i}[\varphi_i(\mathbf{x},\omega)]$$

and

$$f_{\lambda}(\mathbf{x},\mathbb{Q}) := \sum_{i=1}^{n} \lambda_i \mathbb{E}_{\mathbb{Q}_i}[\varphi_i(\mathbf{x},\omega)].$$

We assume that

$$\partial f_{\lambda,\mathcal{Q}}(\mathbf{x}) = \operatorname{conv}\left\{ \bigcup_{\mathbb{Q}\in\mathcal{Q}^*(\mathbf{x})} \partial f_{\lambda}(\mathbf{x},\mathbb{Q}) \right\}, \tag{8}$$

where

$$\begin{aligned} \mathcal{Q}^*(\mathbf{x}) &= \left\{ \bar{\mathbb{Q}} \in \mathcal{Q} : \sup_{\mathbb{Q}\in\mathcal{Q}} f_{\lambda}(\mathbf{x},\mathbb{Q}) = f_{\lambda}(\mathbf{x},\bar{\mathbb{Q}}) \right\} \\ &= \operatorname{argmax}\left\{ f_{\lambda}(\mathbf{x},\mathbb{Q}) : \mathbb{Q} \in \mathcal{Q} \right\}. \end{aligned}$$

For results on the subdifferential of the supremum of general convex functions, we refer to (Zalinescu, 2002, Theorem 2.4.18). In particular, condition 12 implicitly assumes that the convexification introduced in (8) is already closed.

Theorem 13 *Under Assumptions 6 and 12,*

- **(a)** *a point* $\bar{\mathbf{x}} \in \mathcal{X}$ *is a robust weak efficient solution of problem* (RSO$_{\mathcal{Q}}$) *only if there exist* $\bar{k}$ *vectors of probability distributions* $\bar{\mathbb{Q}}^{(1)}, \ldots, \bar{\mathbb{Q}}^{(\bar{k})}$ *all belonging to* $\mathcal{Q}$*, such that* $\bar{\mathbf{x}}$ *is a weakly efficient solution of* $\mathbf{F}_{\bar{\mathbb{Q}}}(\mathcal{X})$.
- **(b)** *a point* $\bar{\mathbf{x}} \in \mathcal{X}$ *is a robust strict efficient solution of problem* (RSO$_{\mathcal{Q}}$) *if, and only if, there exist* $\bar{k}$ *vectors of probability distributions* $\bar{\mathbb{Q}}^{(1)}, \ldots, \bar{\mathbb{Q}}^{(\bar{k})}$ *all belonging to* $\mathcal{Q}$*, such that* $\bar{\mathbf{x}}$ *is a strict efficient solution of the problem of* $\mathbf{F}_{\bar{\mathbb{Q}}}(\mathcal{X})$.

Proof. It follows from Lemma 10 that $A \not\prec^u_{\mathbb{R}^d_+} B$ is equivalent to

$$\exists\, w \equiv w_{A,B} \in \mathbb{S}^1_d : \ \sup_{a\in A}\langle w, a\rangle \geq \sup_{b\in B}\langle w, b\rangle, \tag{9}$$

which implies

$$\sup_{w\in\mathbb{S}^1_d} \left[\sup_{a\in A}\langle w, a\rangle - \sup_{b\in B}\langle w, b\rangle \right] \geq 0; \tag{10}$$

similarly, $A \not\preceq^u_{\mathbb{R}^d_+} B$ is equivalent to

$$\exists\, w \equiv w_{A,B} \in \mathbb{S}^1_d : \ \sup_{a\in A}\langle w, a\rangle > \sup_{b\in B}\langle w, b\rangle,$$

i.e.,

$$\sup_{w\in\mathbb{S}^1_d}\left[\sup_{a\in A}\langle w,a\rangle-\sup_{b\in B}\langle w,b\rangle\right]>0. \tag{11}$$

Let now $\bar{\mathbf{x}}\in\mathcal{X}$. For any $w\in\mathbb{S}^1_d$, we define the following quantity:

$$\gamma_{w,0}:=\sup_{\mathbb{Q}\in\mathcal{Q}}\langle w,g(\bar{\mathbf{x}},\mathbb{Q})\rangle,$$

and the functions $\bar{\gamma}_{w,0}:\mathcal{X}\times\mathcal{Q}\to\mathbb{R}^d,\ \gamma_{w,0}:\mathcal{X}\to\mathbb{R}^d$

$$\begin{aligned}\bar{\gamma}_{w,0}(\mathbf{x},\mathbb{Q}) &:= \langle w,g(\mathbf{x},\mathbb{Q})\rangle-\gamma_{w,0},\\ \gamma_{w,0}(\mathbf{x}) &:= \sup_{\mathbb{Q}\in\mathcal{Q}}\langle w,g(\mathbf{x},\mathbb{Q})\rangle-\sup_{\mathbb{Q}\in\mathcal{Q}}\langle w,g(\bar{\mathbf{x}},\mathbb{Q})\rangle=\sup_{\mathbb{Q}\in\mathcal{Q}}\langle w,g(\mathbf{x},\mathbb{Q})\rangle-\gamma_{w,0}\\ &= \sup_{\mathbb{Q}\in\mathcal{Q}}\left[\langle w,g(\mathbf{x},\mathbb{Q})\rangle-\gamma_{w,0}\right]=\sup_{\mathbb{Q}\in\mathcal{Q}}\bar{\gamma}_{w,0}(\mathbf{x},\mathbb{Q}).\end{aligned}$$

Moreover, we introduce the function $\gamma_0:\mathcal{X}\to\mathbb{R}^d$

$$\gamma_0(\mathbf{x}):=\sup_{w\in\mathbb{S}^1_d}\gamma_{w,0}(\mathbf{x}).$$

It is immediate to see that $\gamma_0(\bar{\mathbf{x}})=0$. Furthermore, taking into account (10) and (11), we obtain that $\bar{\mathbf{x}}$ is a robust weak efficient solution of the robust problem (according to (9)) only if

$$\gamma_0(\mathbf{x})\geq 0,\qquad \forall\mathbf{x}\in\mathcal{X},$$

i.e., only if $\bar{\mathbf{x}}$ is a global minimum of the function $\gamma_0(\mathbf{x})$. Similarly, $\bar{\mathbf{x}}$ is a robust strict efficient solution of the robust problem if, and only if,

$$\gamma_0(\mathbf{x})>0,\qquad \forall\mathbf{x}\in\mathcal{X},$$

i.e., if and only if $\bar{\mathbf{x}}$ is the unique global minimum of $\gamma_0(\mathbf{x})$. Convexity of $\gamma_0(\mathbf{x})$ is ensured under convexity of $\varphi(\mathbf{x},\omega)$ w.r.t. $\mathbf{x}$ (see, e.g. Crespi et al. (2017)). Hence, a necessary condition for optimality in problem (9) is

$$0\in\partial\gamma_0(\bar{\mathbf{x}})+N_{\mathcal{X}}(\bar{\mathbf{x}}).$$

Let $\mathcal{S}^1(\mathbf{x}):=\operatorname{argmax}\left\{\gamma_{w,0}(\mathbf{x}),w\in\mathbb{S}^1_d\right\}$. Under assumption 12,

$$\partial\gamma_0(\bar{\mathbf{x}})=\operatorname{conv}\left\{\partial\gamma_{w,0}(\bar{\mathbf{x}}),w\in\mathcal{S}^1(\bar{\mathbf{x}})\right\},$$

and

$$\partial\gamma_{w,0}(\bar{\mathbf{x}})=\operatorname{conv}\left\{\partial\bar{\gamma}_{w,0}(\bar{\mathbf{x}},\mathbb{Q}):\mathbb{Q}\in\mathcal{Q}_w(\bar{\mathbf{x}})\right\},$$

where

$$\mathcal{Q}_w(\bar{\mathbf{x}})=\operatorname{argmax}\left\{\bar{\gamma}_{w,0}(\bar{\mathbf{x}},\mathbb{Q}),\mathbb{Q}\in\mathcal{Q}\right\}.$$

First, we observe that $\mathbf{z} \in \partial\bar{\gamma}_{w,0}(\bar{\mathbf{x}})$ if and only if there exists $(\mathbb{Q}^{(j)})_{j\in J} \subseteq \mathcal{Q}_w(\bar{\mathbf{x}})$ (with $J \subset \mathbb{N}, \#J < \infty$) such that

$$\mathbf{z} = \sum_{j\in J} t_j \mathbf{z}_j, \qquad \text{with } t_j \geq 0,\ \mathbf{z}_j \in \partial\bar{\gamma}_{w,0}(\bar{\mathbf{x}}, \mathbb{Q}^{(j)})\ \forall\ j \in J \text{ and } \sum_{j\in J} t_j = 1.$$

By linearity of $\gamma_{w,0}(\bar{\mathbf{x}}, \mathbb{Q})$ w.r.t. $\mathbb{Q}$ it follows that $\mathbf{z} \in \partial\gamma_{w,0}(\bar{\mathbf{x}}, \bar{\mathbb{Q}})$ with $\bar{\mathbb{Q}} = \sum_{j\in J} t_j \mathbb{Q}^{(j)}$.

Second, we observe that $\mathbf{v} \in \partial\gamma_0(\bar{\mathbf{x}})$ if and only if there exists $(w_k)_{k\in K} \subseteq \mathcal{S}^1(\bar{\mathbf{x}})$ (with $K \subset \mathbb{N}, \#K = \bar{k} < \infty$), such that

$$\mathbf{v} = \sum_{k\in K} \delta_k \mathbf{v}_k, \quad \delta_k \geq 0, \text{ with } \mathbf{v}_k \in \partial\gamma_{w_k,0}(\bar{\mathbf{x}})\ \forall\ k \in K \text{ and } \sum_{k\in K} \delta_k = 1.$$

Hence, for each $\mathbf{v}_k, k \in K$ such that $\mathbf{v}_k \in \partial\gamma_{w_k,0}(\bar{\mathbf{x}})$ there exist $\mathbf{v}_{kj}$, $(\mathbb{Q}^{(kj)})_{j\in J_k} \subseteq \mathcal{Q}$ and t_{kj} $(j \in J_k)$ such that

$$\mathbf{v}_k = \sum_{j\in J} t_{kj} \mathbf{v}_{kj}, \quad t_{kj} \geq 0, \sum_{j\in J_k} t_{kj} = 1, \text{ with } \mathbf{v}_{kj} \in \partial\gamma_{w_k,0}(\bar{\mathbf{x}}, \mathbb{Q}^{(kj)})\ \forall\ j \in J_k.$$

Hence, $\mathbf{v} = \sum_{k\in K} \delta_k \mathbf{v}_k$ with $\mathbf{v}_k \in \partial\gamma_{w_k,0}(\bar{\mathbf{x}}, \bar{\mathbb{Q}}_k)$ and $\bar{\mathbb{Q}}^{(k)} = \sum_{j\in J} t_{kj} \mathbb{Q}^{(kj)}$. We obtain

$$\mathbf{v} \in \delta_1 \partial\gamma_{w_1,0}(\bar{\mathbf{x}}, \bar{\mathbb{Q}}^{(1)}) + \delta_2 \partial\gamma_{w_2,0}(\bar{\mathbf{x}}, \bar{\mathbb{Q}}^{(2)}) + \cdots + \delta_{\bar{k}} \partial\gamma_{w_{\bar{k}},0}(\bar{\mathbf{x}}, \bar{\mathbb{Q}}^{(\bar{k})}).$$

Thus if $\bar{\mathbf{x}}$ is a minimizer of γ_0 then it is a minimizer of the function

$$\begin{aligned} \ell(\mathbf{x}) &= \delta_1 \gamma_{w_1,0}(\mathbf{x}, \bar{\mathbb{Q}}_1) + \delta_2 \gamma_{w_2,0}(\mathbf{x}, \bar{\mathbb{Q}}_2) + \cdots + \delta_{\bar{k}} \gamma_{w_{\bar{k}},0}(\mathbf{x}, \bar{\mathbb{Q}}_{\bar{k}}) \\ &= \langle \delta_1 w_1, g(\mathbf{x}, \bar{\mathbb{Q}}_1)\rangle + \langle \delta_2 w_2, g(\mathbf{x}, \bar{\mathbb{Q}}_2)\rangle + \cdots + \langle \delta_{\bar{k}} w_{\bar{k}}, g(\mathbf{x}, \bar{\mathbb{Q}}_{\bar{k}})\rangle - \gamma_{w,0} \end{aligned}$$

with $w = \sum_{i=1}^{\bar{k}} \delta_i w_i$ and $\gamma_{w,0} = \sup_{\mathbb{Q}\in\mathcal{Q}} \langle w, g(\mathbf{x}_0, \mathbb{Q})\rangle$. Since $\delta_i w_i \in \mathbb{R}^d_+$ for all $i = 1, \ldots, \bar{k}$, it follows that, if $\bar{\mathbf{x}}$ is a minimizer (resp. the unique minimizer) of γ_0, then it is a weak (resp. strict) efficient solution of the multiobjective optimization problem corresponding to $\mathbf{F}_{\bar{\mathbb{Q}}}$ with $n\bar{k}$ objectives components and ordering cone $\mathbb{R}^{d\bar{k}}_+$. ■

Remark 14 We already know that, if $\bar{\mathbf{x}} \in \mathcal{X}$ is a weak efficient solution of ($\text{OWC}_{\mathcal{C}}$), then $\bar{\mathbf{x}}$ is a weak robust efficient solution of ($\text{RSO}_{\mathcal{Q}}$) (we refer to Proposition 9). Theorems 13 reflects this relation.

3 Link with set-valued risk measures

In this section we investigate the link between the robust counterpart of the uncertain multiobjective optimization problem examined in Section 2.2, namely

$$\min_{\mathbf{x}\in\mathcal{X}} \left\{ \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x}, \omega)] : \mathbb{Q} \in \mathcal{Q} \right\},$$

and the minimization of coherent set-valued risk measures as those introduces by Jouini et al. (2004),Cascos and Molchanov (2007), Hamel and Heyde (2010). We prove, in particular, for a specific class of set-valued risk measures, the two approaches are equivalent, so that the results presented in Section 2.2 can be applied to the minimization of composite set-valued risk functions.

3.1 Set-valued risk measures: definition

In what follows, we introduce the concept of a set-valued risk measure, following the approach in Hamel and Heyde (2010) and Cascos and Molchanov (2007). However, unlike Hamel and Heyde (2010), we consider risk measures that take values in the space of lower closed convex sets, which are partially ordered using the relation $\preceq^u_{\mathbb{R}^d_+}$.

Let $M \subset \mathbb{R}^d$ be a linear subspace with dimension $m \geq 1$. Two important cases are when $m = 1$ and when $m = d$. The reason for introducing M is that an investor or regulator may only accept risk compensations or collateral in a specific subset of the d available markets or currencies. For example, if only the first m components are accepted, then $M = \mathbb{R}^m \times \{0\}^{d-m}$ (see Jouini et al. (2004), Feinstein and Rudloff (2015), Hamel et al. (2011)). We also assume that $M \cap \mathbb{R}^d_+ \neq \{0\}$, meaning that there is at least one nonnegative position in the accepted instruments that can be used as risk compensation or collateral.

By

$$\mathbb{F}_M := \left\{ D \subseteq M : D = \mathrm{cl}(D + (-M \cap \mathbb{R}^d_+)) \right\}$$

we denote the collections of the lower closed convex subsets of M. The $+$ sign denotes the usual Minkowsky addition with with $\emptyset + D = D + \emptyset = \emptyset$ for all $D \subseteq M$. The multiplication is extended by $t\emptyset = \emptyset$ for $t > 0$ and $0 \cdot D = M \cap \mathbb{R}^d_+$ for all $D \in \mathbb{F}_M$; in particular, $0\emptyset = M \cap \mathbb{R}^d_+$.

Definition 15 (See (Hamel and Heyde, 2010, Definition 2.1) and (Cascos and Molchanov, 2007, Definition 2.3)) *A set-valued risk measure is a set-valued function $\mathbf{R} : L^\infty_d \to \mathbb{F}_M$ which is*

(N) *normalized: $\mathbf{R}(0) \subseteq -M \cap \mathbb{R}^d_+$ and $\mathbf{R}(0) \cap (\mathrm{int}\, M \cap \mathbb{R}^d_+) = \emptyset$;*

(T) *translative: for all $\mathbf{X} \in L^\infty_d$ and all $u \in M$,*

$$\mathbf{R}(\mathbf{X} + u\mathbb{1}) = \mathbf{R}(\mathbf{X}) - u;$$

(M) *monotone: for all $\mathbf{X}, \mathbf{Y}$ such that $\mathbf{X} \leq_{\mathbb{R}^d_+} \mathbf{Y}$, $\mathbb{P}$-a.s., it holds $\mathbf{R}(\mathbf{Y}) \preceq^u_{\mathbb{R}^d_+} \mathbf{R}(\mathbf{X})$;*

(CVX) *convex: for all $\mathbf{X}, \mathbf{Y} \in L^\infty_d$ and $\lambda \in (0,1)$,*

$$\mathbf{R}(\lambda \mathbf{X} + (1-\lambda)\mathbf{Y}) \subseteq \lambda \mathbf{R}(\mathbf{X}) + (1-\lambda)\mathbf{R}(\mathbf{Y}), \qquad \textit{or, equivalently,}$$
$$\mathbf{R}(\lambda \mathbf{X} + (1-\lambda)\mathbf{Y}) \preceq^u_{\mathbb{R}^d_+} \lambda \mathbf{R}(\mathbf{X}) + (1-\lambda)\mathbf{R}(\mathbf{Y}).$$

If $\mathbf{R}$ *satisfies (N),(T), (M) and (CVX), then it is called a set-valued convex measure of risk.*

The value $\mathbf{R}(\mathbf{X})$ represents all the possible vectors of accepted reference instruments that can be used to compensate for the risk of the multivariate position $\mathbf{X}$. In other words, it describes the set of eligible risk compensations for $\mathbf{X}$. For a more in-depth discussion of the financial meaning of $\mathbf{R}$, we refer the reader to (Hamel and Heyde, 2010, Section 2).

3.2 Link between set-valued risk measure minimization and robust set-optimization

Now we set $M = \mathbb{R}^d$ and we consider the function $\mathbf{R} : L_d^\infty \to \mathbf{F}_M$ defined as

$$\mathbf{R}(\mathbf{X}) := \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+, \tag{12}$$

where $\mathcal{Q} \subseteq (\mathcal{M}_{1,d}(\mathbb{P}))^d$.

We prove that this is a convex set-valued function in the sense of Definition 15.

Theorem 16 *The set-valued functional defined in* (12) *is a convex set-valued risk measure.*

Proof. (N) $\mathbf{R}(0) = \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[0] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+ = -\mathbb{R}^d_+$. This proves normalization.

(T) Let $u \in \mathbb{R}^d$. Then

$$\begin{aligned}\mathbf{R}(\mathbf{X}+u) &= \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}-u] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+ \\ &= \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}] - u : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+ \\ &= \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+ - u \\ &= \mathbf{R}(\mathbf{X}) - u\end{aligned}$$

We conclude that $\mathbf{R}$ is translative.

(M) Let $\mathbf{X}, \mathbf{Y} \in L_d^\infty$ such that $\mathbf{X} \leq_{\mathbb{R}^d_+} \mathbf{Y}$ $\mathbb{P}$-a.s. Then $X_i \leq Y_i$ for all $i = 1, \ldots, d$ and, consequently, for any $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d) \in \mathcal{Q}$ we also have

$$\mathbb{E}_{\mathbb{Q}_i}[-X_i] \geq \mathbb{E}_{\mathbb{Q}_i}[-Y_i], \quad i = 1, \ldots, d.$$

This implies that $\mathbb{E}_{\mathbb{Q}}[-\mathbf{Y}] \leq_{\mathbb{R}^d_+} \mathbb{E}_{\mathbb{Q}}[-\mathbf{X}]$. Thus

$$\begin{aligned}\mathbf{R}(\mathbf{Y}) = \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{Y}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+ \\ \subseteq \operatorname{cl}\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+ = \mathbf{R}(\mathbf{X}),\end{aligned}$$

i.e., $\mathbf{R}(\mathbf{Y}) \preceq^u_{\mathbb{R}^d_+} \mathbf{R}(\mathbf{X})$. This proves monotonicity with respect to the set order relation $\preceq^u_{\mathbb{R}^d_+}$.

(CVX) Define

$$\widetilde{\mathbf{R}}(\mathbf{X}) := \operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+.$$

Then,

$$\widetilde{\mathbf{R}}(\lambda\mathbf{X}+(1-\lambda)\mathbf{Y}) = \operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\lambda\mathbf{X} - (1-\lambda)\mathbf{Y}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+,$$

while

$$\begin{aligned} &\lambda\widetilde{\mathbf{R}}(\mathbf{X}) + (1-\lambda)\widetilde{\mathbf{R}}(\mathbf{Y}) \\ &= \lambda\left(\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+\right) + (1-\lambda)\left(\operatorname{co}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{Y}] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^d_+\right). \end{aligned}$$

Suppose that $u \in \widetilde{\mathbf{R}}(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y})$. Then there exist $v \in \mathbb{R}^d_+$, $\bar{\mathbb{Q}}^1, \ldots, \bar{\mathbb{Q}}^n \in \mathcal{Q}$ and $\delta_j \in (0,1)$, $j = 1, \ldots, n$ with $\sum_{j=1}^n \delta_j = 1$ such that

$$u = \sum_{j=1}^{n} \mathbb{E}_{\bar{\mathbb{Q}}^j}[-\lambda\mathbf{X} - (1-\lambda)\mathbf{Y}] - v.$$

Notice that

$$\begin{aligned} u &= \lambda\left(\sum_{j=1}^{n} \delta_j \mathbb{E}_{\bar{\mathbb{Q}}^j}[-\mathbf{X}] - v\right) + (1-\lambda)\left(\sum_{j=1}^{n} \delta_j \mathbb{E}_{\bar{\mathbb{Q}}^j}[-\mathbf{Y}] - v\right) \\ &=: \lambda w_{\mathbf{X}} + (1-\lambda) w_{\mathbf{Y}}, \end{aligned}$$

with $w_{\mathbf{X}} \in \widetilde{\mathbf{R}}(\mathbf{X}), w_{\mathbf{Y}} \in \widetilde{\mathbf{R}}(\mathbf{Y})$. We conclude that $\widetilde{\mathbf{R}}(\lambda\mathbf{X}+(1-\lambda)\mathbf{Y}) \subseteq \lambda\widetilde{\mathbf{R}}(\mathbf{X})+(1-\lambda)\widetilde{\mathbf{R}}(\mathbf{Y})$, or, equivalently,

$$\widetilde{\mathbf{R}}(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) \preceq^u_{\mathbb{R}^d_+} \lambda\widetilde{\mathbf{R}}(\mathbf{X}) + (1-\lambda)\widetilde{\mathbf{R}}(\mathbf{Y}).$$

The convexity of $\mathbf{R}$ is now readily obtained by passing to the closures of the aforementioned sets.

■

Remark 17 *Given a function $\psi : \mathbb{R}^k \to L^\infty_d$ and a set-valued risk measure $\mathbf{R}$ with values in $\mathbf{F}_{\mathbb{R}^d}$, the corresponding composite risk function is $\mathbf{R}(\psi(\mathbf{x}))$, and the associated minimization problem is*

$$\begin{aligned} &\text{minimize} \;\; \mathbf{R}(\psi(\mathbf{x})), \\ &\text{s.t.} \;\; \mathbf{x} \in \mathcal{X}, \end{aligned}$$

where the minimization is understood with respect to a given set order relation, in our case $\preceq^u_{\mathbb{R}^d_+}$.

It is worth noting that composite set-valued functions of the form $\mathbf{R}\circ\psi$ inherit convexity properties under suitable assumptions. Specifically, if ψ is concave with respect to the partial order $\leq_{\mathbb{R}^d_+}$—that is,

$$\psi(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \geq_{\mathbb{R}^d_+} \alpha\psi(\mathbf{x}_1) + (1-\alpha)\psi(\mathbf{x}_2), \qquad \mathbb{P}\text{-}a.s.,$$

and if $\mathbf{R}$ *is a monotone and convex set-valued risk measure, then the composition* $\mathbf{R} \circ \psi$ *is convex as a function from* $\mathbb{R}^n$ *into* $\mathbb{F}_M$*. This follows from the fact that convexity of* $\mathbf{R}$ *implies*

$$\begin{aligned}\alpha \mathbf{R}(\psi(\mathbf{x}_1)) + (1-\alpha)\mathbf{R}(\psi(\mathbf{x}_2)) &= \alpha \mathbf{R}(\mathbf{X}^1) + (1-\alpha)\mathbf{R}(\mathbf{X}^2) \\ &\supseteq \mathbf{R}(\alpha \mathbf{X}^1 + (1-\alpha)\mathbf{X}^2) = \mathbf{R}(\alpha\psi(\mathbf{x}_1) + (1-\alpha)\psi(\mathbf{x}_2)),\end{aligned}$$

where $\mathbf{X}^1 = \psi(\mathbf{x}_1)$ *and* $\mathbf{X}^2 = \psi(\mathbf{x}_2)$*. Monotonicity of* $\mathbf{R}$ *then ensures that*

$$\begin{aligned}\alpha \mathbf{R}(\psi(\mathbf{x}_1)) + (1-\alpha)\mathbf{R}(\psi(\mathbf{x}_2)) &\supseteq \mathbf{R}(\alpha\psi(\mathbf{x}_1) + (1-\alpha)\psi(\mathbf{x}_2)) \\ &\supseteq \mathbf{R}(\psi(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2)),\end{aligned}$$

where the last inclusion follows from the assumed concavity of ψ*. This chain of inclusions confirms the convexity of the composite function* $\mathbf{R} \circ \psi$*.*

Notice that the form of the composite risk function is analogous to the function $\mathbf{F}_{\mathcal{Q}}$ *introduced in* (RSO$_{\mathcal{Q}}$)*.*

4 An illustrative example

Multicriteria optimization is a well-established framework to analyze portfolio optimization problems in the sense of Markowitz. We work in a financial market with d risky assets with uncertain outcomes $X_1, \dots, X_d$; $\boldsymbol{\theta} \in \mathbb{R}^d$ is a portfolio vector representing the proportions of wealth invested in the d assets. Following Section 2, we consider two minimization problems.

4.1 Set-valued risk measures

Suppose $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega_1\}) = p$, $\mathbb{P}(\{\omega_2\}) = 1-p$ with $0 < p < 1$ and $d = 2$. Any measure $\mathbb{Q} \ll \mathbb{P}$ is represented by a vector $\mathbf{q} = (q, 1-q) \in \mathbb{R}^2$ with $q \in (0,1)$. Hence, any vector of probability measures absolutely continuous with respect to $\mathbb{P}$ is described by a pair of measures

$$\mathbb{Q} = \begin{pmatrix} \mathbb{Q}_1 \\ \mathbb{Q}_2 \end{pmatrix} \qquad \text{with} \qquad \mathbb{Q}_i = (q_i, 1-q_i), \quad q_i \in (0,1), \quad i = 1,2.$$

Suppose that $\mathcal{Q} = \operatorname{clco}\left\{\mathbb{Q}^1, \mathbb{Q}^2\right\}$ for some $\mathbb{Q}^1, \mathbb{Q}^2 \ll \mathbb{P}$. In the following we will use the following notation to indicate each component of $\mathbb{Q}^j, j = 1,2$:

$$\mathbb{Q}^j = \begin{pmatrix} \mathbb{Q}^j_1 \\ \mathbb{Q}^j_2 \end{pmatrix} = \begin{pmatrix} q^j_1 & 1-q^j_1 \\ q^j_2 & 1-q^j_2 \end{pmatrix}, \qquad j = 1,2.$$

Moreover we observe that $\mathcal{Q}$ can be described as

$$\begin{aligned}\mathcal{Q} &= \left\{\alpha \mathbb{Q}^1 + (1-\alpha)\mathbb{Q}^2 : \alpha \in [0,1]\right\} \\ &= \left\{ \begin{pmatrix} \alpha \mathbb{Q}^1_1 + (1-\alpha)\mathbb{Q}^2_1 \\ \alpha \mathbb{Q}^1_2 + (1-\alpha)\mathbb{Q}^2_2 \end{pmatrix} : \alpha \in [0,1] \right\}.\end{aligned}$$

Any r.v. $\mathbf{X} = (X_1, X_2)^\top$ on Ω is represented by a pair of 2-dimensional vectors

$$\mathbf{X}(\omega_1) = \begin{pmatrix} X_1(\omega_1) \\ X_2(\omega_1) \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{X}(\omega_2) \quad = \begin{pmatrix} X_1(\omega_2) \\ X_2(\omega_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Now assume that $\mathbf{X}$ is fixed and that $\boldsymbol{\theta} = (\theta_1, \theta_2)$ represents the vector of weights of our portfolio. Set $\mathbf{X}(\boldsymbol{\theta}) = (\theta_1 X_1, \theta_2 X_2)$. Suppose that we want to minimize the set-valued risk measure

$$\mathbf{R}(\mathbf{X}(\boldsymbol{\theta})) := \operatorname{clco}\{\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}(\boldsymbol{\theta})] : \mathbb{Q} \in \mathcal{Q}\} - \mathbb{R}^2_+.$$

We write explicitly the expression of $\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}(\boldsymbol{\theta})]$ for any $\mathbb{Q} \in \mathcal{Q}$. Setting $\mu^i_j = \mathbb{E}_{\mathbb{Q}^i_j}[X_j]$, $i = 1, 2$, $j = 1, 2$, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[-\mathbf{X}(\boldsymbol{\theta})] &= \begin{bmatrix} \mathbb{E}_{\mathbb{Q}_1}[-\theta_1 X_1] \\ \mathbb{E}_{\mathbb{Q}_2}[-\theta_2 X_2] \end{bmatrix} = \begin{bmatrix} \mathbb{E}_{\alpha\mathbb{Q}^1_1 + (1-\alpha)\mathbb{Q}^2_1}[-\theta_1 X_1] \\ \mathbb{E}_{\alpha\mathbb{Q}^1_2 + (1-\alpha)\mathbb{Q}^2_2}[-\theta_2 X_2] \end{bmatrix} \\
&= \begin{bmatrix} -\theta_1 \mathbb{E}_{\alpha\mathbb{Q}^1_1 + (1-\alpha)\mathbb{Q}^2_1}[X_1] \\ -\theta_2 \mathbb{E}_{\alpha\mathbb{Q}^1_2 + (1-\alpha)\mathbb{Q}^2_2}[X_2] \end{bmatrix} \\
&= \begin{bmatrix} -\theta_1(\alpha \mathbb{E}_{\mathbb{Q}^1_1}[X_1] + (1-\alpha)\mathbb{E}_{\mathbb{Q}^2_1}[X_1]) \\ -\theta_2(\alpha \mathbb{E}_{\mathbb{Q}^1_2}[X_2] + (1-\alpha)\mathbb{E}_{\mathbb{Q}^2_2}[X_2]) \end{bmatrix} \\
&= -\alpha \operatorname{diag}(\boldsymbol{\theta})\, \boldsymbol{\mu}^1 - (1-\alpha)\operatorname{diag}(\boldsymbol{\theta})\boldsymbol{\mu}^2 \\
&= -\alpha p_1(\boldsymbol{\theta}) - (1-\alpha) p_2(\boldsymbol{\theta}),
\end{aligned}$$

where

$$\boldsymbol{\mu}^j = \begin{bmatrix} \mu^j_1 \\ \mu^j_2 \end{bmatrix}, \qquad \text{and} \qquad p_j(\boldsymbol{\theta}) = \operatorname{diag}(\boldsymbol{\theta})\boldsymbol{\mu}^j, \qquad j = 1, 2.$$

Hence

$$\mathbf{R}(\mathbf{X}(\boldsymbol{\theta})) = \{-\alpha p_1(\boldsymbol{\theta}) - (1-\alpha)p_2(\boldsymbol{\theta}) : \alpha \in [0,1]\} - \mathbb{R}^2_+.$$

Our set-minimization problem is

$$\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathbf{R}(\mathbf{X}(\boldsymbol{\theta})), \tag{P}$$

w.r.t. the set order relation $\preceq^u_{\mathbb{R}^d_+}$.

Now, following Section 2.2, we fix $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, $v \in \mathbb{S}^1_2$, and define

$$\begin{aligned}
g(\boldsymbol{\theta}, \alpha) &= -\alpha p_1(\boldsymbol{\theta}) - (1-\alpha)p_2(\boldsymbol{\theta}), \quad \alpha \in [0,1],\ \boldsymbol{\theta} \in \boldsymbol{\Theta}, \\
\gamma_{v,0} &= \sup_{\alpha \in [0,1]} \langle v, g(\boldsymbol{\theta}_0, \alpha) \rangle, \\
\gamma_{v,0}(\boldsymbol{\theta}) &= \sup_{\alpha \in [0,1]} (\langle v, g(\boldsymbol{\theta}, \alpha) \rangle - \gamma_{v,0}), \quad \boldsymbol{\theta} \in \boldsymbol{\Theta}, \\
\gamma_0(\boldsymbol{\theta}) &= \sup_{v \in \mathbb{S}^1} \sup_{\alpha \in [0,1]} (\langle v, g(\boldsymbol{\theta}, \alpha) \rangle - \gamma_{v,0}), \quad \boldsymbol{\theta} \in \boldsymbol{\Theta}.
\end{aligned}$$

Notice that $\gamma_{v,0}$ can be written also as

$$\gamma_{v,0}(\boldsymbol{\theta}) = \sup_{\mathbf{x} \in \mathbf{R}(\mathbf{X}(\boldsymbol{\theta}))} (\langle v, \mathbf{x} \rangle - \gamma_{v,0}).$$

Robust solutions can be now found applying Theorem 13; in particular $\boldsymbol{\theta}_0$ is a strict robust solution of problem (P) if and only if it is the unique minimizer of γ_0.

We consider two reference vectors $\boldsymbol{\mu}^1 = [1,2]^\top$ and $\boldsymbol{\mu}^2 = [3,1]^\top$, and define $p_j(\boldsymbol{\theta}) = \mathrm{diag}(\boldsymbol{\theta})\mu^j$ for $j = 1,2$. The set-valued risk measure $\mathbf{R}(\mathbf{X}(\boldsymbol{\theta}))$ is given by the convex combinations of $-p_1(\boldsymbol{\theta})$ and $-p_2(\boldsymbol{\theta})$, extended by subtracting $\mathbb{R}^2_+$ to obtain a lower closed set.

Figure 1 shows the sets $\mathbf{R}(\mathbf{X}(\boldsymbol{\theta})) - \mathbb{R}^2_+$ for three different values of $\boldsymbol{\theta}$: $\boldsymbol{\theta}_1 = [0.5, 0.5]$ (red), $\boldsymbol{\theta}_2 = [0.2, 0.8]$ (green), and the optimal $\boldsymbol{\theta}_0$ (blue), which is the unique minimizer of the support function $\gamma_{v,0}(\boldsymbol{\theta})$ in direction $v = [1,2]$. The shaded regions represent the lower closed extensions of the convex combinations, and the optimal set corresponds to the minimal value of the support function in the given direction.

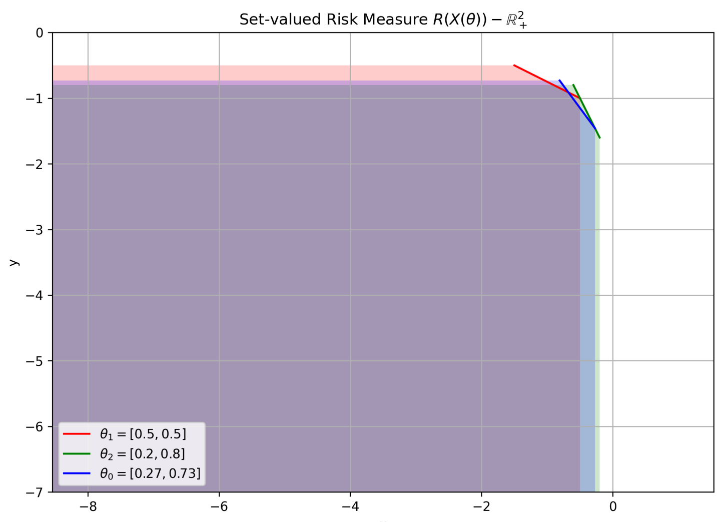

Figure 1: Set-valued risk measure $\mathbf{R}(\mathbf{X}(\boldsymbol{\theta}))$ with lower closed regions for three different values of $\boldsymbol{\theta}$.

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