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Multi-objective optimization and its connection to multivariate risk measures

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Abstract In this paper, we generalize the study of minimax stochastic programming to the case where the objective function is multi-objective. We adopt a component-wise worst-case approach and provide necessary and sufficient conditions for optimality in terms of suitable first-order conditions. We then compare the proposed method with the minimization of vector-valued risk measures, as developed progressively in the literature over the past decades. We show that minimizing a certain class of multivariate risk measures is, in a precise sense, equivalent to solving a multi-objective expected value optimization problem with respect to some appropriate admissible distributions. We also analyze specific optimization problems involving risk functionals.

Keywords stochastic programming · minimax robustness · worst-case · risk minimization

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1 Introduction

Let us consider a stochastic system whose output is represented by an n-dimensional random vector defined on a measurable space $(\Omega, \mathcal{F})$. If the system depends on a decision variable $\mathbf{x} \in \mathbb{R}^k$, the corresponding outcome can be expressed as

$$\varphi(\mathbf{x}, \omega) \in \mathbb{R}^n, \quad \omega \in \Omega$$

where the mapping $\varphi \colon V \times \Omega \to \mathbb{R}^n$ is assumed to be measurable, with

$$V \subseteq \mathbb{R}^k.$$

When needed, we denote the random vector associated with a fixed decision $\mathbf{x}$ by

$$\mathbf{X}(\omega) \coloneqq \varphi(\mathbf{x}, \omega),$$

although we primarily work with the functional form φ. In the present work, we focus on the case where $\varphi(\mathbf{x}, \cdot)$ models the uncertain value of a portfolio of financial positions, whose realization depends on the underlying scenario $\omega \in \Omega$.

To determine the optimal decision vector $\mathbf{x}$, a standard strategy (see, e.g., [6], [28], [29]) is to minimize the expected value of φ with respect to a given vector of probability measures

$$\mathbb{P} = (\mathbb{P}_1, \ldots, \mathbb{P}_n)$$

on $(\Omega, \mathcal{F})$, while ensuring feasibility of the decision variable. This leads to the following multi-objective problem:

$$\min_{\mathbf{x} \in \mathcal{X}} \mathbf{f}_{\mathbb{P}}(\mathbf{x}), \tag{MOP$_{\mathbb{P}}$}$$

where

$$\mathcal{X} \subseteq V$$

and $\mathbf{f}_{\mathbb{P}} \equiv \mathbf{f}_{\mathbb{P},\varphi} \colon V \to \mathbb{R}^n$ is the vector-valued objective function defined componentwise as

$$\mathbf{f}_{\mathbb{P}}(\mathbf{x}) \coloneqq \begin{bmatrix} \mathbb{E}_{\mathbb{P}_1}[\varphi_1(\mathbf{x}, \omega)] \\ \vdots \\ \mathbb{E}_{\mathbb{P}_n}[\varphi_n(\mathbf{x}, \omega)] \end{bmatrix}. \tag{1}$$

Problem (MOP$_{\mathbb{P}}$) constitutes a multi-objective stochastic optimization problem. In contrast to scalar settings, the notion of minimization requires careful specification. In financial applications, minimizing expected losses often yields optimal decisions only in *average* terms, potentially overlooking tail risks or distributional asymmetries. Moreover, assuming full knowledge of the vector $(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ is a strong simplification and rarely reflects realistic data-driven scenarios.

To overcome these limitations, various robust strategies have been proposed. In the one-dimensional case (i.e., when $\mathbf{f}_{\mathbb{P}}$ is real-valued), one common approach is to identify a plausible nonempty family $\mathcal{Q}$ of probability distributions, and replace the original expected value—under a single probability measure $\mathbb{P}$—with its worst-case *counterpart*:

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x},\omega)]$$

(where φ is scalar-valued). This leads to the *robust* formulation:

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x},\omega)].$$

This model, investigated by several authors including [4], guarantees a worst-case performance over all admissible distributions. Alternative robust paradigms have also been proposed; see, for instance, [5], [12], [15].

The main goal of this paper is to extend this robust framework to the multi-objective setting of (MOP$_{\mathbb{P}}$), linking it to vector-valued risk minimization as introduced in [7] (see also [19], [21]). Our analysis builds on techniques and results from [13,22], and adopts a component-wise robustification strategy.

For clarity of exposition, we initially focus on the canonical ordering cone

$$K = \mathbb{R}^n_+ := [0,\infty[^n,$$

before generalizing to more abstract cones and topological structures in Section 2 and Section 3.

(C-RC) Component-wise robust counterpart of (MOP$_{\mathbb{P}}$). *Let $\mathcal{Q}$ denote a class of vectors of probability measures containing $\mathbb{P}$. The objective is to determine the efficient points of the function $\mathbf{f}_{\mathcal{C}}\colon V \to \mathbb{R}^n$, defined as*

$$\mathbf{f}_{\mathcal{C}}(\mathbf{x}) := \begin{bmatrix} \sup_{\mathbb{Q}_1\in\mathcal{C}_1} \mathbb{E}_{\mathbb{Q}_1}[\varphi_1(\mathbf{x},\omega)] \\ \vdots \\ \sup_{\mathbb{Q}_n\in\mathcal{C}_n} \mathbb{E}_{\mathbb{Q}_n}[\varphi_n(\mathbf{x},\omega)] \end{bmatrix}, \tag{2}$$

over the feasible set $\mathcal{X}$, where each $\mathcal{C}_i$ denotes the projection onto the i-th component of the convex hull

$$\mathcal{C} := \operatorname{co}\mathcal{Q}.$$

The corresponding robust optimization problem reads:

$$\min_{\mathbf{x}\in\mathcal{X}} \mathbf{f}_{\mathcal{C}}(\mathbf{x}). \tag{3}$$

Further discussion of the topological and probabilistic framework adopted in this setting is provided in Section 2, particularly in Subsection 2.2.

We show that, under suitable assumptions, there exists an n-dimensional vector of distributions $\bar{\mathbb{Q}} = \left(\bar{\mathbb{Q}}_1,\dots,\bar{\mathbb{Q}}_n\right) \in \mathcal{C}$ such that solving (3) reduces to identifying the efficient set of the multi-objective function $\mathbf{f}_{\bar{\mathbb{Q}}}$ over $\mathcal{X}$—that is, the function $\mathbf{f}_{\mathbb{P}}$ with $\mathbb{P} = \bar{\mathbb{Q}}$, as in (1).

We also derive necessary and sufficient optimality conditions for pairs $(\bar{\mathbf{x}}, \bar{\mathbb{Q}})$ via subdifferential analysis of suitable scalarizations.

In the second part of the paper, we explore the relationship between the component-wise robust model and vector-valued risk minimization. These involve quantifying risk through measures that summarize loss distributions over a given horizon. Examples include vector-valued extensions of classical risk measures such as Value-at-Risk and Expected Shortfall (see [21], [7], [19]). The associated minimization problem is formulated as follows.

(RMv) Risk minimization problem in the vector setting. *Let $\psi\colon \mathcal{X} \to \mathcal{Y}$ be a measurable mapping into a measurable space $\mathcal{Y}$ of n-dimensional random vectors, and let $\boldsymbol{\varrho}\colon \mathcal{Y} \to \mathbb{R}^n$ be a vector-valued risk measure in the sense of [7]. We aim to determine the efficient points of*

$$\boldsymbol{\varrho}\big(\psi(\mathbf{x})\big),$$

over the feasible region $\mathcal{X}$.

We provide conditions under which the composition $\boldsymbol{\varrho} \circ \psi$ is convex, and show the equivalence between problem **(RMv)** and a subclass of robust problems of type **(C-RC)**. A numerical example in Section 5 illustrates the practical implications of our framework.

The remainder of the paper is structured as follows. Section 2 introduces the essential machinery from vector optimization. Section 3 focuses on robust multi-objective stochastic optimization problems and characterization of solutions by means of scalarizations and their subdifferentials. In particular, we discuss the objective-wise worst-case approach. Section 4 explores connections of **(C-RC)** with vector-valued risk minimization, including dual representations. Section 5 provides a numerical case study in portfolio optimization. Finally, Section 6 outlines our conclusions and directions for future research.

2 Basic notions and definitions

In this section, we recall fundamental notions and notational conventions from multi-objective optimization theory, which will be used throughout the paper—particularly in Section 3. We also review basic concepts from probability theory that are essential for the developments in the second part of the paper.

2.1 Multi-objective conic optimization

Let $k, n \geq 1$ be two fixed positive integers. Multi-objective optimization concerns the task of minimizing, in an appropriate sense, a vector-valued function

$$\mathbf{f} = (f_1, \ldots, f_n)^\top\colon \mathcal{X} \to \mathbb{R}^n$$

subject to constraints that define the feasible set $\mathcal{X} \subseteq \mathbb{R}^k$, as in (MOP$_\mathbb{P}$) and (3). Since the image space $\mathbf{f}(\mathcal{X}) \subseteq \mathbb{R}^n$ of $\mathbf{f}$ generally lacks a canonical total order, the notion of minimization requires a rigorous specification.

To this purpose, let $K \subseteq \mathbb{R}^n$ be an arbitrary closed, convex, and pointed cone with nonempty interior cone. The vector partial order $\leq_K$ induced by K on $\mathbb{R}^n$ is then defined by: for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$, $\mathbf{z}_1 \leq_K \mathbf{z}_2$ (or $\mathbf{z}_2 \geq_K \mathbf{z}_1$) whenever

$$\mathbf{z}_2 - \mathbf{z}_1 \in K$$

(geometrically, $\mathbf{z}_1$ dominates $\mathbf{z}_2$ in all admissible directions defined by K). This binary relation is a preorder—reflexive and transitive—antisymmetric by pointedness, and compatible with the algebraic structure of $\mathbb{R}^n$.

As is commonly done in the literature on this subject, the cone K is assumed to contain the nonnegative orthant:

$$\mathbb{R}^n_+ \subseteq K,$$

a condition which, in turn, constrains its dual K^* to lie within $\mathbb{R}^n_+$. Recall that the dual cone K^* of K is defined as

$$K^* \coloneqq \big\{\mathbf{w} \in \mathbb{R}^n : \forall\, \mathbf{z} \in K,\ \langle \mathbf{w},\mathbf{z}\rangle \geq 0\big\}, \tag{4}$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes, here and throughout the manuscript, the standard Euclidean inner product. Under our assumptions, the cone K^* also has nonempty interior.

In what follows, we shall adopt the notational conventions

$$<_K \coloneqq \leq_{\operatorname{int} K}, \qquad \lneq_K \coloneqq \leq_{K\setminus\{0\}}$$

and, in analogy, the standard component-wise orderings induced by $\mathbb{R}^n_+$:

$$\leq \coloneqq \leq_{\mathbb{R}^n_+}, \qquad < \coloneqq \leq_{\operatorname{int} \mathbb{R}^n_+}, \qquad \lneq \coloneqq \leq_{\mathbb{R}^n_+\setminus\{0\}}.$$

It should be noted that these satisfy the following implication chain:

$$<_K \implies \lneq_K \implies \leq_K .$$

We introduce below three categories of efficient solutions, ordered from the weakest to the strongest form of "K-efficiency" they exhibit (see, e.g., [16]).

Definition 2.1 (Efficiency notions) Given a multi-objective optimization problem similar to those previously introduced, a feasible solution $\bar{\mathbf{x}} \in \mathcal{X}$ is said to be, with respect to K:

- *weakly efficient* if there exists no $\mathbf{x} \in \mathcal{X}$ such that

$$\mathbf{f}(\mathbf{x}) <_K \mathbf{f}(\bar{\mathbf{x}});$$

- *efficient* if there exists no $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ such that

$$\mathbf{f}(\mathbf{x}) \lneq_K \mathbf{f}(\bar{\mathbf{x}});$$

– *properly efficient* if it is efficient, and there exists a convex, pointed, and closed cone $\widetilde{K} \subseteq \mathbb{R}^n$, with $K \subseteq \widetilde{K}$, for which there exists no $\mathbf{x} \in \mathcal{X} \setminus \{\bar{\mathbf{x}}\}$ such that
$$\mathbf{f}(\mathbf{x}) \leq_{\widetilde{K}} \mathbf{f}(\bar{\mathbf{x}}).$$

Put simply, a point is weakly efficient if no other point dominates it in all components; others may outperform it in some, but not across the board. A point is efficient if no alternative is at least as good in all dimensions and strictly better in one—improving any part inevitably means sacrificing another. A properly efficient point limits how much one component can improve without a steep loss elsewhere, ensuring more balanced trade-offs.

Each of these notions reflects a different degree of *nondominance* under the partial order $\leq_K$. For an arbitrary subset of $\mathbb{R}^n$, the collection of its K-efficient (or nondominated) points is indicated by
$$\leq_K - \inf.$$
Similarly,
$$\leq_K - \sup$$
is regarded as the set of K-dominating elements. Despite the symbols used, it is important to emphasize that these concepts are substantially distinct from those of infimum and supremum with respect to $\leq_K$ (of a subset of $\mathbb{R}^n$).

In multi-criteria optimization (e.g., cost, efficiency, sustainability), computing the conic infimum $\leq_K - \inf$ means identifying all solutions that cannot be improved along any admissible direction—those in K—without simultaneously worsening at least one criterion. These K-efficient points thus represent the nondominated outcomes of the problem.

Remark 2.1 In the case where $\mathcal{X}$ is convex and $\mathbf{f}$ is convex on $\mathcal{X}$, a point $\bar{\mathbf{x}} \in \mathcal{X}$ is properly efficient (with respect to K) if, and only if, it is efficient, and there exists a vector $\mathbf{w} \in \operatorname{int} K^*$ (see (4)) such that
$$\bar{\mathbf{x}} \in \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}} \langle \mathbf{w}, \mathbf{f}(\mathbf{x}) \rangle.$$

In what follows, we focus on a K-minimization problem in $\mathbb{R}^n$ whose objective function departs from standard multi-criteria formulations. Specifically, it is defined via conic suprema over a family of vector-valued mappings—a structure that naturally emerges in robust optimization settings.

This formulation is motivated by the need to incorporate robustness into the model, in light of multiple sources of uncertainty or variability in the data. The aim is to identify solutions that remain nondominated (with respect to K) under worst-case realizations within the prescribed uncertainty set Ω. This naturally leads to objective functions constructed through sup operations, capturing the robustified counterpart of a standard multi-criteria objective.

More precisely, the conic infimum (respectively, supremum) of a $\mathbb{R}^n$-valued function is understood as the infimum (respectively, supremum) of its image set, with respect to the partial order $\leq_K$, in the case where it corresponds to a unique vector rather than a possibly infinite set.

In this regard, we introduce the notations

$$K-\mathrm{GLB}, \qquad K-\mathrm{LUB} \tag{5}$$

to represent the infimum and supremum, respectively, with respect to the partial order $\leq_K$ of a subset of $\mathbb{R}^n$—terms that evocatively stand for "greatest lower bound" and "least upper bound" with respect to K, respectively. Although non-standard, these symbols serve to clearly distinguish the notions of efficiency from those of infimum and supremum, specifically in contexts where they coincide with single vectors, not sets (as opposed to, for instance, the K-efficient frontier itself). This approach will be employed sparingly, as we will establish immediately below that, under suitable geometric assumptions on K, our analysis can be reduced to the canonical case where

$$K = \mathbb{R}^n_+,$$

at which stage the use of traditional symbols for infimum and supremum, instead of those in (5), will present no ambiguity.

Indeed, when the cone K is *polyhedral*, [8, Section 3] ensures that optimizing a sufficiently regular vector-valued function with respect to K, in the sense of the efficient frontier, is equivalent to *Pareto* optimizing the function obtained by applying an appropriate linear mapping (to the earlier function).

At this point, if K is spanned by an invertible square matrix, then the standard minimization problem **(C-RC)**, namely (3), as proposed in the Introduction, admits a fairly intuitive extension that aligns with the structure induced by K, which we will define in (11) below. Since the objective function (2) is a supremum with respect to the nonnegative orthant, i.e., an $\mathbb{R}^n_+ -\mathrm{LUB}$ (componentwise suprema arise naturally), the adaptation of (2) will, in turn, after a linear transformation, become precisely a $K-\mathrm{LUB}$, provided by (10)—that is, (9), up to a change of basis. This juncture is pivotal in our work.

In particular, for a K of this nature, we can always refer to the K–LUB of any given subset of $\mathbb{R}^n$ as its unique pointwise supremum, owing to its intrinsic connection with the nonnegative orthant—an essential simplification that facilitates tractable dual representations and concrete optimization results. This aspect is further discussed in greater detail in Remark 2.4 below.

Therefore, by definition and in this context, the $K-\mathrm{LUB}$ of a set of n-dimensional vectors with respect to the conic order $\leq_K$ admits a powerful algebraic characterization: it solves a non-singular linear square system. Indeed, since K is the cone spanned by the n columns of A, the supremum lies at the intersection of hyperplanes orthogonal to its n rows.

In words, this supremum is the smallest element—under the order $\leq_K$—that still dominates all others. It is the minimal point that remains greater than or equal to all others along the directions defined by K; from a geometric standpoint, it lies "above" the entire set in the cone's partial order.

Considering all this, we shall henceforth restrict our attention exclusively to the case where K is a pointed polyhedral cone generated by a non-singular square matrix, that is, a matrix-cone; this assumption shall remain in force throughout the paper. Accordingly, let

$$A \equiv A_K = (a_{i,j})_{i,j=1,\dots,n}$$

be any invertible matrix in $\mathbb{R}^{n\times n}$ such that

$$K = A^{-1}(\mathbb{R}^n_+) \equiv \Big\{ \mathbf{z} \in \mathbb{R}^n : A\mathbf{z} \in \mathbb{R}^n_+ \Big\}. \tag{6}$$

Moreover, for each $i = 1, \dots, n$, let A_i denote the i-th row of A.

The cone K is closed, convex, contains the origin, and has nonempty interior. The partial order $\leq_K$ introduced earlier now admits the following equivalent characterizations: for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^n$,

$$\mathbf{z}_1 \leq_K \mathbf{z}_2 \quad \Longleftrightarrow \quad A \cdot (\mathbf{z}_2 - \mathbf{z}_1) \in \mathbb{R}^n_+ \quad \Longleftrightarrow \quad A\mathbf{z}_1 \leq A\mathbf{z}_2\,.$$

The inclusion of strictly positive directions ensures that K is full-dimensional, and that the matrix A consists of nonnegative entries; that is, for all $i, j = 1, \dots, n$,

$$a_{i,j} \geq 0\,. \tag{7}$$

Building on this, for any $\mathbb{Q} = (\mathbb{Q}_1, \dots, \mathbb{Q}_n) \in \mathcal{C}$, consider the component-wise expectation map $\mathbf{f}_{\mathbb{Q}} \colon \mathcal{X} \to \mathbb{R}^n$, as defined in (1) (with $\mathbb{P} = \mathbb{Q}$): namely,

$$\mathbf{f}_{\mathbb{Q}}(\mathbf{x}) \equiv \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x},\omega)] \coloneqq \begin{bmatrix} \mathbb{E}_{\mathbb{Q}_1}[\varphi_1(\mathbf{x},\omega)] \\ \vdots \\ \mathbb{E}_{\mathbb{Q}_n}[\varphi_n(\mathbf{x},\omega)] \end{bmatrix}. \tag{8}$$

Then, as previously mentioned, for each $\mathbf{x} \in \mathcal{X}$, the value of the *cone robustification* function

$$K\text{--}\mathrm{LUB}_{\mathbb{Q}\in\mathcal{C}}\, \mathbf{f}_{\mathbb{Q}}(\mathbf{x})$$

coincides with the unique solution $\bar{\mathbf{z}}_{\mathbf{x}} \in \mathbb{R}^n$ of the $n \times n$ linear system

$$A\,\bar{\mathbf{z}}_{\mathbf{x}} = \mathbf{f}_{\mathcal{C},A}(\mathbf{x}),$$

where $\mathbf{f}_{\mathcal{C},A} \colon \mathcal{X} \to \mathbb{R}^n$ is given by

$$\mathbf{f}_{\mathcal{C},A}(\mathbf{x}) \coloneqq \begin{bmatrix} \sup_{\mathbb{Q}^{(1)}\in\mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(1)}}(\mathbf{x}), A_1 \rangle \\ \vdots \\ \sup_{\mathbb{Q}^{(n)}\in\mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(n)}}(\mathbf{x}), A_n \rangle \end{bmatrix}, \quad \mathbf{x} \in \mathcal{X} \tag{9}$$

($\bar{\mathbf{z}}_{\mathbf{x}}$ corresponds to the unique intersection point of n independent affine hyperplanes in $\mathbb{R}^n$ whose normal vectors are $A_1, \dots, A_n$).

Explicitly, for any $\mathbf{x} \in \mathcal{X}$,

$$K\text{--}\mathrm{LUB}_{\mathbb{Q}\in\mathcal{C}}\, \mathbf{f}_{\mathbb{Q}}(\mathbf{x}) = A^{-1}\,\mathbf{f}_{\mathcal{C},A}(\mathbf{x}). \tag{10}$$

Observe that, for any $i = 1, \dots, n$, $\mathbb{Q} \in \mathcal{C}$, and $\mathbf{x} \in \mathcal{X}$, the identity

$$\langle \mathbf{f}_{\mathbb{Q}}(\mathbf{x}), A_i \rangle = \big(A\,\mathbf{f}_{\mathbb{Q}}(\mathbf{x})\big)_i$$

holds (where the right-hand side denotes the i-th component of the matrix-vector product).

Now, consider the associated optimization problem:

$$\leq_K - \min_{\mathbf{x}\in\mathcal{X}} K\!-\!\mathrm{LUB}_{\mathbb{Q}\in\mathcal{C}}\, \mathbf{f}_{\mathbb{Q}}(\mathbf{x}) \tag{11}$$

(optimization is performed with respect to the partial order induced by K). By [8, Section 3], problem (11) is equivalent to the Pareto minimization problem:

$$\leq - \min_{\mathbf{x}\in\mathcal{X}} \mathbf{f}_{\mathcal{C},A}(\mathbf{x}) \tag{$A-\mathrm{OWC}_{\mathcal{C}}$}$$

(as follows from (10)). In this reformulated version, the inverse of matrix A is no longer explicitly required.

We emphasize that, for any $i = 1, \dots, n$ and $\mathbf{x} \in \mathcal{X}$,

$$\begin{aligned}
(\mathbf{f}_{\mathcal{C},A})_i(\mathbf{x}) &= \sup_{\mathbb{Q}^{(i)}\in\mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\mathbf{x}), A_i\rangle \\
&= \sup_{\mathbb{Q}^{(i)}\in\mathcal{C}} \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}_j^{(i)}}[a_{i,j}\varphi_j(\mathbf{x},\omega)]
\end{aligned} \tag{12}$$

(see (8) and (9)). Therefore, despite the absence of A^{-1}, problem ($A-\mathrm{OWC}_{\mathcal{C}}$) does not retain a component-wise structure in the strict sense.

Remark 2.2 Once more by [8, Section 3],

$$\leq_K - \min_{\mathbf{x}\in\mathcal{X}} \mathbf{f}_{\mathcal{C}}(\mathbf{x}) \qquad \Longleftrightarrow \qquad \leq - \min_{\mathbf{x}\in\mathcal{X}} A\,\mathbf{f}_{\mathcal{C}}(\mathbf{x}) \tag{13}$$

(refer to (2) and (3)). In particular, the resulting Pareto minimization problem in (13) no longer exhibits a purely component-wise structure. Indeed, for any $\mathbf{x} \in \mathcal{X}$,

$$A\,\mathbf{f}_{\mathcal{C}}(\mathbf{x}) = \begin{bmatrix} \langle \mathbf{f}_{\mathcal{C}}(\mathbf{x}), A_1\rangle \\ \vdots \\ \langle \mathbf{f}_{\mathcal{C}}(\mathbf{x}), A_n\rangle \end{bmatrix}, \tag{14}$$

i.e., more explicitly, the i-th component of $A\,\mathbf{f}_{\mathcal{C}}(\mathbf{x})$ reads as

$$\sum_{j=1}^{n} a_{i,j} \sup_{\mathbb{Q}_j\in\mathcal{C}_j} \mathbb{E}_{\mathbb{Q}_j}[\varphi_j(\mathbf{x},\omega)] = \sum_{j=1}^{n} \sup_{\mathbb{Q}_j\in\mathcal{C}_j} \mathbb{E}_{\mathbb{Q}_j}[a_{i,j}\varphi_j(\mathbf{x},\omega)]. \tag{15}$$

As a consequence, expressions (12) and (15) coincide when the product set

$$\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$$

is considered in place of the convex hull $\mathcal{C}$. Under this specification, it follows that, for any $\mathbf{x} \in \mathcal{X}$,

$$K\!-\!\mathrm{LUB}_{\mathbb{Q}\in\mathcal{C}}\, \mathbf{f}_{\mathbb{Q}}(\mathbf{x}) = \mathbf{f}_{\mathcal{C}}(\mathbf{x})$$

(see (14)), and therefore, the three minimization problems (11), ($A-\mathrm{OWC}_{\mathcal{C}}$), and (13), become equivalent. This equivalence also holds, notably, when A is a diagonal matrix:

$$A = \begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{bmatrix}.$$

According to [10, Chapter 3], weakly and properly efficient solutions, defined with respect to the classical Pareto ordering induced by $K = \mathbb{R}^n_+$, can be characterized in terms of optimal solutions to *scalarized problems*, in which the objective function is expressed as a weighted-sum of the original components, with nonnegative (respectively, strictly positive) weights. This yields a family of *scalarized objective functions* naturally associated with the original vector optimization problem.

In view of the equivalence established above between problems (11) and ($A-\mathrm{OWC}_{\mathcal{C}}$), this classical scalarization framework remains fully applicable in our setting as well. As a consequence, scalarization plays a central role in identifying Pareto-type optima, as summarized in the following fundamental results.

Proposition 2.1 (See [10, Propositions 3.9 and 3.10]) *Let $\bar{\mathbf{x}} \in \mathcal{X}$ be an optimal solution of the weighted-sum optimization problem*

$$\min_{\mathbf{x} \in \mathcal{X}} \sum_{i=1}^{n} \lambda_i f_i(\mathbf{x}), \tag{16}$$

where $\lambda := (\lambda_1, \ldots, \lambda_n)^\top \in \mathbb{R}^n_+$. If

$$\lambda \in \mathbb{R}^n_+ \setminus \{0\},$$

then $\bar{\mathbf{x}}$ is a weakly efficient point of ($\mathrm{MOP}_{\mathbb{P}}$) *with respect to $K = \mathbb{R}^n_+$.*

Moreover, if $\mathcal{X}$ is convex and each f_i is a convex function ($i = 1, \ldots, n$), then $\bar{\mathbf{x}}$ is weakly efficient with respect to $\mathbb{R}^n_+$ (if and) only if there exists $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\bar{\mathbf{x}}$ solves (16).

Proposition 2.2 (See [10, Theorem 3.11 and 3.15], or [16]). *Let $\lambda \in \operatorname{int} \mathbb{R}^n_+$. If $\bar{\mathbf{x}} \in \mathcal{X}$ is an optimal solution of* (16)*, then $\bar{\mathbf{x}}$ is properly efficient for* ($\mathrm{MOP}_{\mathbb{P}}$) *with respect to $K = \mathbb{R}^n_+$. In addition, if $\mathcal{X}$ is convex and each f_i is convex, then $\bar{\mathbf{x}}$ is properly efficient with respect to $\mathbb{R}^n_+$ (if and) only if there exists $\lambda \in \operatorname{int} \mathbb{R}^n_+$ such that $\bar{\mathbf{x}}$ solves* (16).

Remark 2.3 For notational convenience, the symbol 0—in plain font—is used throughout to denote the zero vector $\mathbf{0}$ as well, whether deterministic or random (irrespective of its dimension or probabilistic structure).

If the optimization problem is ($A-\mathrm{OWC}_{\mathcal{C}}$), that is, if $\mathbf{f} = \mathbf{f}_{\mathcal{C},A}$ is as defined in (9), then the scalarized objective function associated with a generic weight vector $(\lambda_1, \ldots, \lambda_n)^\top$ is, for any $\mathbf{x} \in \mathcal{X}$, given by

$$\sum_{i=1}^{n} \lambda_i \sup_{\mathbb{Q}^{(i)} \in \mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\mathbf{x}), A_i \rangle = \sum_{i=1}^{n} \sup_{\mathbb{Q}^{(i)} \in \mathcal{C}} \lambda_i \sum_{j=1}^{n} \mathbb{E}_{\mathbb{Q}_j^{(i)}}[a_{i,j} \varphi_j(\mathbf{x}, \omega)]. \tag{17}$$

For instance, when $n = 2$, the scalarized objective function takes the form

$$\sup_{(\mathbb{Q}_1^{(1)}, \mathbb{Q}_2^{(1)}) \in \mathcal{C}} \lambda_1 \left\{ \mathbb{E}_{\mathbb{Q}_1^{(1)}}[a_{1,1} \varphi_1(\mathbf{x}, \omega)] + \mathbb{E}_{\mathbb{Q}_2^{(1)}}[a_{1,2} \varphi_2(\mathbf{x}, \omega)] \right\}$$
$$+ \sup_{(\mathbb{Q}_1^{(2)}, \mathbb{Q}_2^{(2)}) \in \mathcal{C}} \lambda_2 \left\{ \mathbb{E}_{\mathbb{Q}_1^{(2)}}[a_{2,1} \varphi_1(\mathbf{x}, \omega)] + \mathbb{E}_{\mathbb{Q}_2^{(2)}}[a_{2,2} \varphi_2(\mathbf{x}, \omega)] \right\}.$$

Remark 2.4 Assuming that A is square and invertible allows the problem to be addressed in a genuinely vector-valued manner. In this case—and only in this case—the K-supremum of a set reduces to a single vector, enabling the framework to circumvent the analytical complexities inherent in set-valued formulations.

In contrast, when A is not invertible or not square—even if it remains of full rank (i.e., with trivial kernel)—the supremum must instead be interpreted as a set of dominating points, thus requiring more general and sophisticated tools. Several approaches have been proposed in the literature to address this need.

- In [10] and [11], scalarization-based strategies are used to reformulate multi-criteria problems as parametric families of scalar ones, preserving tractability even when the ordering structure lacks full dimensionality. The former offers a foundational treatment, while the latter develops a robust minimax framework that captures directional uncertainty.
- In [17] and [18], a lattice-theoretic generalization of the supremum is outlined by viewing $\mathbb{R}^n$ as a conlinear space ordered by K, where the supremum of a set is defined as the intersection of all closed, K-upper sets containing it. This formulation ensures the existence of a least upper set even when the cone is not generated by a basis.
- In [19], a set-valued duality framework, designed for coherent risk measures with non-singleton outputs, is constructed via the conic closure and convex hull of the risk function's epigraph. This enables the definition of generalized suprema compatible with convex duality, even in the absence of structural regularity in the matrix A.

These alternative formulations suggest promising directions for extending the current vector-based framework to more general, possibly degenerate, ordering cones. We leave their development for future research.

2.2 Preliminaries on probability spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We denote by

$$\mathcal{M}_{1,f}(\mathbb{P})$$

the set of all (σ-additive) probability measures $\mathbb{Q}$ on $\mathcal{F}$ that are absolutely continuous with respect to $\mathbb{P}$, meaning that $\mathbb{P}(E) = 0$ implies $\mathbb{Q}(E) = 0$ for all $E \in \mathcal{F}$. For any $\mathbb{Q} \in \mathcal{M}_{1,f}(\mathbb{P})$, the expression $d\mathbb{Q}/d\mathbb{P}$ represents for the equivalence class of Radon-Nikodým derivatives of $\mathbb{Q}$ with respect to $\mathbb{P}$: that is,

$$\mathbb{Q}(E) = \int_E \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\, d\mathbb{P}(\omega), \quad E \in \mathcal{F}.$$

The space $L^\infty_n(\Omega, \mathcal{F}, \mathbb{P})$, also written as $L^\infty_n(\mathbb{P})$ or L^∞_n, is the Banach space of all equivalence classes of essentially bounded $\mathbb{R}^n$-valued random variables $\mathbf{X} = (X_1, \ldots, X_n)^\top$, i.e., those such that

$$\|\mathbf{X}\|_\infty \coloneqq \operatorname{ess.sup} |\mathbf{X}| < \infty,$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. In the scalar case ($n = 1$), we simply write $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, or more briefly $L^\infty(\mathbb{P})$ and L^∞.

We shall refer to $L^\infty_{n,+}(\mathbb{P})$, or simply $L^\infty_{n,+}$, as the collection of $\mathbb{R}^n_+$-valued random variables in L^∞_n.

Analogously, $L^1_n \coloneqq L^1_n(\Omega, \mathcal{F}, \mathbb{P})$—abbreviated as L^1 if $n = 1$—stands for the (Banach) space of integrable $\mathbb{R}^n$-valued random variables with respect to $\mathbb{P}$. We set $L^1_{n,+} \subseteq L^1_n$ as the cone of $\mathbb{R}^n_+$-valued elements, and $L^1_{n,-} \subseteq L^1_n$ as the cone of $\mathbb{R}^n_-$-valued ones, where $\mathbb{R}^n_- \coloneqq \left]-\infty, 0\right]^n$. These spaces correspond, respectively, to the positive and negative dual directions in the weak*-topology induced by the duality pairing with L^∞_n.

For $X \in L^1$ and $\mathbb{Q} \in \mathcal{M}_{1,f}(\mathbb{P})$, the symbol $\mathbb{E}_\mathbb{Q}[X]$ is used to indicate the expected value (i.e., the integral) of X with respect to $\mathbb{Q}$.

We write

$$\mathrm{ba}_n(\mathbb{P})$$

for the space of bounded finitely additive set-valued functions $\mu = (\mu_1, \ldots, \mu_n)$ on $(\Omega, \mathcal{F})$ whose components are absolutely continuous with respect to $\mathbb{P}$.

This space, endowed with the total variation norm

$$\|\mu\| \coloneqq \sup \left\{ \sum_{i=1}^m |\mu(E_i)| : E_1, \ldots, E_k \text{ disjoint in } \mathcal{F} \right\},$$

is isometrically isomorphic to the dual of L^∞_n. It thus induces—on $\mathrm{ba}_n(\mathbb{P})$—the weak* topology

$$\sigma\big(\mathrm{ba}_n(\mathbb{P}), L^\infty_n\big).$$

For any $\mathbf{X} \in L^\infty_n$, the action of $\mu \in ba_n(\mathbb{P})$ on $\mathbf{X}$ is given by

$$\langle \mu, \mathbf{X} \rangle \coloneqq \sum_{j=1}^n \mathbb{E}_{\mu_j}[X_j] = \sum_{j=1}^n \int_\Omega X_j(\omega)\, \mathrm{d}\mu_j(\omega),$$

with a slight abuse of notation when using the expectation symbol for measures not necessarily probabilistic.

Countably additive elements of $\mathrm{ba}_n(\mathbb{P})$ can be identified with elements in L^1_n. Moreover, any subset $\mathcal{Q} \subseteq (\mathcal{M}_{1,f}(\mathbb{P}))^n$ may also be regarded as a subset of $\mathrm{ba}_n(\mathbb{P})$ via the canonical embedding.

For each component μ_i of any $\mu \in \mathrm{ba}_n(\mathbb{P})$, we retain the same notation for the Radon-Nikodým derivative of μ_i with respect to $\mathbb{P}$ as is employed for the elements in $\mathcal{M}_{1,f}(\mathbb{P})$.

Further details on these functional-analytic foundations may be found in [21].

3 Robust objective-wise worst-case approach

As motivated in the Introduction, many practical problems inherently involve uncertainty: key input data—such as the expected future returns of random variables—are not precisely known and must be estimated. This naturally gives rise to multi-objective optimization problems with uncertain parameters.

In this section, we extend the classical scalar minimax stochastic optimization problem

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x},\omega)]$$

to a multi-objective framework, and analyze its fundamental theoretical properties. To define the component-wise robust counterpart of the multi-objective optimization problem associated with the function above, we proceed essentially along the lines of [13], which we briefly summarize below.

Consider the family of optimization problems parametrized by vectors of probability measures $\mathbb{Q} \in \mathcal{Q}$:

$$\leq_K - \min_{\mathbf{x}\in\mathcal{X}} \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x},\omega)], \qquad (\mathrm{MOP}_{\mathbb{Q}})$$

where the minimization is taken with respect to the partial order induced by the cone K (see (8)). We assume that the following conditions hold.

Assumption 3.1

Assumption on $\mathcal{X}$:

(A0) $\mathcal{X}$ is a nonempty, closed, and convex subset of $\mathbb{R}^k$.

Assumptions on φ_i, $i = 1,\dots,n$: there exists a convex neighborhood V of $\mathcal{X}$—that is, a convex set containing an open neighborhood of every point in $\mathcal{X}$—such that $\varphi = (\varphi_1,\dots,\varphi_n)^\top$ is an $\mathbb{R}^n$-valued measurable mapping defined on $V \times \Omega$, and the following three properties are satisfied.

(A1) For all $\mathbf{x} \in V$ and $i = 1,\dots,n$, the function $\varphi_i(\mathbf{x},\cdot)$ belong to $L^\infty(\mathbb{P})$.
(A2) For every $\omega \in \Omega$ and $i = 1,\dots,n$, the function $\varphi_i(\cdot,\omega)$ is convex on V.

(A3) For all $i = 1, \ldots, n$, the sup-function

$$(\mathbf{f}_{\mathcal{C},A})_i(\cdot) = \sup_{\mathbb{Q}^{(i)} \in \mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\cdot), A_i \rangle$$

(see (12)) is finite valued; that is, for any $\mathbf{x} \in V$, it holds that $(\mathbf{f}_{\mathcal{C},A})_i(\mathbf{x}) \in \mathbb{R}$.

Assumption on $\mathcal{Q}$:

(A4) $\mathcal{Q}$ is a nonempty and weak*-closed subset of $\mathrm{ba}_n(\mathbb{P})$.

For all $i = 1, \ldots, n$, let $f_i \colon V \times \mathcal{C} \to \mathbb{R}$ be defined as

$$f_i(\mathbf{x}, \mathbb{Q}) \coloneqq \langle \mathbf{f}_{\mathbb{Q}}(\mathbf{x}), A_i \rangle, \quad (\mathbf{x}, \mathbb{Q}) \in V \times \mathcal{C} \tag{18}$$

(see (8)), so that, for any $i = 1, \ldots, n$ and $\mathbf{x} \in \mathcal{X}$,

$$(\mathbf{f}_{\mathcal{C},A})_i(\mathbf{x}) = \sup_{\mathbb{Q}^{(i)} \in \mathcal{C}} f_i(\mathbf{x}, \mathbb{Q}^{(i)}). \tag{19}$$

Note that each f_i is also well-defined on the entire space $V \times \mathrm{ba}_n(\mathbb{P})$; see Subsection 2.2. Moreover, for any $\mathbb{Q} \in \mathcal{C}$, the map

$$\mathbf{x} \mapsto f_i(\mathbf{x}, \mathbb{Q})$$

is convex on V—and therefore continuous on (an open neighborhood of) $\mathcal{X}$—and, for any $\mathbf{x} \in V$, the map

$$\mathbb{Q} \mapsto f_i(\mathbf{x}, \mathbb{Q})$$

is affine with respect to convex combinations in $\mathcal{C}$ (note also (7)). In particular, since the component-wise suprema in $\mathbf{f}_{\mathcal{C},A}$ (see (9)) are invariant under convexification, the function could equivalently be defined over the original set $\mathcal{Q}$ rather than over its convex hull

$$\mathcal{C} = \left\{ \mathbb{Q} = \sum_{j \in J} p_j \mathbb{Q}^{(j)} : \mathbb{Q}^{(j)} \in \mathcal{Q},\, p_j > 0,\, \sum_{j \in J} p_j = 1,\, J \in \mathscr{P}_{\mathrm{fin}}(\mathbb{N}) \right\},$$

where $\mathscr{P}_{\mathrm{fin}}(\mathbb{N})$ denotes the family of all finite subsets of $\mathbb{N}$.

We emphasize that $\mathcal{C}$ is a bounded subset of $(\mathcal{M}_{1,f}(\mathbb{P}))^n$ with respect to the L^1_n-norm, and therefore also with respect to the $\mathrm{ba}_n(\mathbb{P})$-norm. Furthermore, since $\mathcal{Q}$ is assumed to be weak*-closed in $\mathrm{ba}_n(\mathbb{P})$, $\mathcal{C}$ is likewise weak*-closed and, ultimately, strongly closed.

We therefore adopt an objective-wise worst-case perspective, replacing the global uncertainty set $\mathcal{Q}$ with the structured set $\mathcal{C}$. For each $i = 1, \ldots, n$, the corresponding marginal projection $\mathcal{C}_i$ is explicitly given by

$$\mathcal{C}_i = \left\{ \tilde{\mathbb{Q}} \in \mathcal{M}_{1,f}(\mathbb{P}) : \exists\ \mathbb{Q} = \big(\mathbb{Q}_1, \ldots, \mathbb{Q}_i, \ldots, \mathbb{Q}_n\big) \in \mathcal{C} \text{ with } \mathbb{Q}_i = \tilde{\mathbb{Q}} \right\}.$$

It should be noted that, for any $\mathbf{x} \in V$, the map $\mathbb{Q} \mapsto f_i(\mathbf{x}, \mathbb{Q})$ is also upper semicontinuous on $\mathcal{C}$; see, e.g., [27].

Mimicking the scalar case, we define the robust counterpart of the family of problems $\mathrm{MOP}_{\mathbb{Q}}$ by adopting an objective-wise worst-case formulation, as in (11):

$$\leq_K - \min_{\mathbf{x}\in\mathcal{X}} K\!-\mathrm{LUB}_{\mathbb{Q}\in\mathcal{C}}\, \mathbb{E}_{\mathbb{Q}}[\varphi(\mathbf{x},\omega)].$$

As pointed out in [11], when dealing with multiple objectives, one cannot simply evaluate a solution by taking a supremum over all probability measures, since the expression $\mathbb{E}_{\mathbb{Q}}[\,\varphi(\mathbf{x},\omega)\,]$ is vector-valued and the notion of “sup” requires a careful and rigorous interpretation in the vector setting.

Problem (11), i.e., $(A\!-\!\mathrm{OWC}_{\mathcal{C}})$, captures this extension by replacing each component of the objective function in $(\mathrm{MOP}_{\mathbb{Q}})$ with its corresponding robust counterpart. This formulation generalizes the approach introduced in [22, p. 307] and [13, Definition 2.7]—originally developed in the context of finite-dimensional uncertainty sets—to the setting of vector-valued expectations over families of measures (see also [3, 20]).

To solve the problem $(A\!-\!\mathrm{OWC}_{\mathcal{C}})$, linear scalarization techniques can be applied. We start by analyzing the properties of the component functions $(\mathbf{f}_{\mathcal{C},A})_i$ (see (12)).

Lemma 3.1 *Let Assumption 3.1 hold. Then, for any $i = 1,\ldots,n$, the function $(\mathbf{f}_{\mathcal{C},A})_i$ is real-valued, lower semicontinuous and convex on $\mathcal{X}$.*

Proof By Assumption 3.1.**(A2)**, each function $(\mathbf{f}_{\mathcal{C},A})_i$, $i = 1,\ldots,n$, is convex on V because it is the supremum of real valued convex functions (see (19)), and it is finite according to 3.1.**(A3)**. Furthermore, these functions are lower semicontinuous on $\mathcal{X}$ due to the property of being the supremum of continuous functions on $\mathcal{X}$ (as discussed, for instance, in [24]). □

The following result gives a characterization of weak and properly efficient points in terms of a saddle point condition. It extends the characterization given in [22, Theorem 2.1. and Theorem 2.3] to the case where the uncertainty is represented by a (possibly infinite-dimensional) set of vector of probability measures.

Recall that a *saddle point* $(\bar{\mathbf{x}}, \underline{\bar{\mathbb{Q}}}) \in V \times \mathcal{C}^n$, with

$$\underline{\bar{\mathbb{Q}}} = \big(\bar{\mathbb{Q}}^{(1)},\ldots,\bar{\mathbb{Q}}^{(n)}\big), \quad \bar{\mathbb{Q}}^{(i)} = \big(\bar{\mathbb{Q}}_1^{(i)},\ldots,\bar{\mathbb{Q}}_n^{(i)}\big) \in \mathcal{C}$$

for all $i = 1,\ldots,n$ (that is, $\underline{\bar{\mathbb{Q}}}$ can be regarded as an $n\times n$ matrix of probability measures belonging to $\mathcal{C}$, in the context of the scalarized function

$$\begin{aligned} \Xi_\lambda(\mathbf{x},\underline{\mathbb{Q}}) &:= \lambda \cdot \begin{bmatrix} \langle \mathbf{f}_{\mathbb{Q}^{(1)}}(\mathbf{x}), A_1\rangle \\ \vdots \\ \langle \mathbf{f}_{\mathbb{Q}^{(n)}}(\mathbf{x}), A_n\rangle \end{bmatrix} \\ &= \sum_{i=1}^n \lambda_i \sum_{j=1}^n \mathbb{E}_{\mathbb{Q}_j^{(i)}}[a_{i,j}\,\varphi_j(\mathbf{x},\omega)], \end{aligned} \tag{20}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)^\top \in \mathbb{R}^n_+$, $\mathbf{x} \in \mathcal{X}$, and

$$\underline{\mathbb{Q}} = \left(\mathbb{Q}^{(1)}, \dots, \mathbb{Q}^{(n)}\right) \in \mathcal{C}^n, \quad \mathbb{Q}^{(i)} = \left(\mathbb{Q}^{(i)}_1, \dots, \mathbb{Q}^{(i)}_n\right) \in \mathcal{C}$$

$(i = 1, \dots, n)$, is characterized by:

$$\begin{cases} \bar{\mathbf{x}} \in \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}} \ \Xi_\lambda(\mathbf{x}, \bar{\underline{\mathbb{Q}}}) \\ \bar{\underline{\mathbb{Q}}} \in \operatorname{argmax}_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \ \Xi_\lambda(\bar{\mathbf{x}}, \underline{\mathbb{Q}}). \end{cases}$$

This saddle-point formulation reflects the robust scalarization structure typically arising in multi-objective optimization under uncertainty; see, e.g., [4, 3, 10].

Theorem 3.2 *Under Assumption 3.1,*

(a) $\bar{\mathbf{x}} \in \mathcal{X}$ *is a weakly efficient solution of problem* $(A-\mathrm{OWC}_\mathcal{C})$ *if, and only if, there exist* $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ *and* $\bar{\underline{\mathbb{Q}}} \in \mathcal{C}^n$ *such that*

$$\begin{cases} \bar{\mathbf{x}} \in \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}} \ \Xi_\lambda(\mathbf{x}, \bar{\underline{\mathbb{Q}}}) \\ \bar{\underline{\mathbb{Q}}} \in \operatorname{argmax}_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \ \Xi_\lambda(\bar{\mathbf{x}}, \underline{\mathbb{Q}}); \end{cases}$$

(b) $\bar{\mathbf{x}} \in \mathcal{X}$ *is a properly efficient solution of problem* $(A-\mathrm{OWC}_\mathcal{C})$ *if, and only if, there exist* $\lambda \in \operatorname{int} \mathbb{R}^n_+$ *and* $\bar{\underline{\mathbb{Q}}} = \left(\bar{\mathbb{Q}}^{(1)}, \dots, \bar{\mathbb{Q}}^{(n)}\right) \in \mathcal{C}^n$ *such that*

$$\begin{cases} \bar{\mathbf{x}} \in \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}} \ \Xi_\lambda(\mathbf{x}, \bar{\underline{\mathbb{Q}}}) \\ \bar{\mathbb{Q}}^{(i)} \in \operatorname{argmax}_{\mathbb{Q}^{(i)}\in\mathcal{C}} \ \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\bar{\mathbf{x}}), A_i \rangle, \quad i = 1, \dots, n. \end{cases} \tag{21}$$

Proof The proof basically requires the use of a minimax theorem for general vector spaces (i.e., non necessarily finite-dimensional).

(a) From Proposition 2.1, a point $\bar{\mathbf{x}}$ is a weakly efficient solution of $(A-\mathrm{OWC}_\mathcal{C})$ if, and only if, there exists $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ such that $\bar{\mathbf{x}}$ is a solution of the weighted-sum scalarization problem

$$\min_{\mathbf{x}\in\mathcal{X}} \sum_{i=1}^n \lambda_i \sup_{\mathbb{Q}^{(i)}\in\mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\mathbf{x}), A_i \rangle = \min_{\mathbf{x}\in\mathcal{X}} \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \sum_{i=1}^n \lambda_i \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\mathbf{x}), A_i \rangle,$$

where

$$\underline{\mathbb{Q}} = \left(\mathbb{Q}^{(1)}, \dots, \mathbb{Q}^{(n)}\right) \in \mathcal{C}^n, \quad \mathbb{Q}^{(i)} = \left(\mathbb{Q}^{(i)}_1, \dots, \mathbb{Q}^{(i)}_n\right) \in \mathcal{C}$$

(see also (9) and (17)). Notice that the equality above is easily verifiable due to the structure of the problem.

Necessity. Suppose $\bar{\mathbf{x}} \in \mathcal{X}$ is an optimal solution of

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \sum_{i=1}^n \lambda_i \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\mathbf{x}), A_i \rangle.$$

Then, by Lemma 3.1, the function $\Xi_\lambda(\mathbf{x}, \underline{\mathbb{Q}})$ is lower semicontinuous with respect to $\mathbf{x} \in \mathcal{X}$ and upper semicontinuous with respect to $\underline{\mathbb{Q}}$ (strongly upper semicontinuous and, consequently, weakly* upper semicontinuous as well). Additionally, Ξ_λ is convex with respect to $\mathbf{x}$ and concave with respect to $\underline{\mathbb{Q}}$; in particular, it is linear with respect to $\underline{\mathbb{Q}}$ when considering $(\mathrm{ba}_n(\mathbb{P}))^n$ as (part of) the domain of Ξ_λ rather than $(\mathcal{M}_{1,f}(\mathbb{P}))^{n\times n}$. We then derive from [30, Theorem 4*] that

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \Xi_\lambda(\mathbf{x},\underline{\mathbb{Q}}) = \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \inf_{\mathbf{x}\in\mathcal{X}} \Xi_\lambda(\mathbf{x},\underline{\mathbb{Q}}).$$

In general, the right-hand side is not (the supremum of) an actual minimum over $\mathbf{x}\in\mathcal{X}$.

Moreover, by assumption, the set $\mathcal{C}$ is bounded, convex, and (weakly*-) closed in $(\mathcal{M}_{1,f}(\mathbb{P}))^n$. By the Banach-Alaoglu theorem, it is weakly* compact in $(\mathcal{M}_{1,f}(\mathbb{P}))^n$. Then (see Remark 2 and Corollary 1 in [30]) we also have that the supremum is realized at some $\bar{\underline{\mathbb{Q}}}\in\mathcal{C}^n$, i.e.,

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \Xi_\lambda(\mathbf{x},\underline{\mathbb{Q}}) = \max_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \inf_{\mathbf{x}\in\mathcal{X}} \Xi_\lambda(\mathbf{x},\underline{\mathbb{Q}})$$

and (see, e.g., [24, Theorem 2]) $(\bar{\mathbf{x}},\bar{\underline{\mathbb{Q}}})$ is a saddle point of Ξ_λ.
Sufficiency. Conversely, if $(\bar{\mathbf{x}},\bar{\underline{\mathbb{Q}}})$ are optimal solutions of

$$\begin{cases} \bar{\mathbf{x}}\in \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}}\ \Xi_\lambda(\mathbf{x},\bar{\underline{\mathbb{Q}}}) \\ \bar{\underline{\mathbb{Q}}}\in \operatorname{argmax}_{\underline{\mathbb{Q}}\in\mathcal{C}^n}\ \Xi_\lambda(\bar{\mathbf{x}},\underline{\mathbb{Q}}), \end{cases}$$

then $(\bar{\mathbf{x}},\bar{\underline{\mathbb{Q}}})$ is a saddle point of Ξ_λ and $\bar{\mathbf{x}}$ is an optimal solution of

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \Xi_\lambda(\mathbf{x},\underline{\mathbb{Q}}).$$

(b) From Proposition 2.2, a point $\bar{\mathbf{x}}$ is a properly efficient solution of $(A-\mathrm{OWC}_{\mathcal{C}})$ if, and only if, there exists $\lambda\in\operatorname{int}\mathbb{R}^n_+$ such that $\bar{\mathbf{x}}$ is a solution of the weighted-sum scalarization problem

$$\min_{\mathbf{x}\in\mathcal{X}} \sup_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \sum_{i=1}^n \lambda_i \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\mathbf{x}),A_i\rangle.$$

Proceeding as above, we can prove that this is equivalent to the existence of some $\bar{\underline{\mathbb{Q}}}=\left(\bar{\mathbb{Q}}^{(1)},\dots,\bar{\mathbb{Q}}^{(n)}\right)\in\mathcal{C}^n$ such that $\bar{\mathbf{x}}$ is a global minimum of

$$\Xi_\lambda(\mathbf{x},\bar{\underline{\mathbb{Q}}})$$

and $\bar{\underline{\mathbb{Q}}}$ is a maximum of

$$\Xi_\lambda(\bar{\mathbf{x}},\underline{\mathbb{Q}}).$$

It remains to prove (21). As observed in [22, Theorem 2.1], this condition easily follows by taking into account that, if $\lambda_i>0$ for all $i=1,\dots,n$, then the standard inequality

$$\langle \mathbf{f}_{\bar{\mathbb{Q}}^{(i)}}(\bar{\mathbf{x}}),A_i\rangle \le \sup_{\mathbb{Q}^{(i)}\in\mathcal{C}} \langle \mathbf{f}_{\mathbb{Q}^{(i)}}(\bar{\mathbf{x}}),A_i\rangle$$

is, actually, an identity $(i=1,\dots,n)$. □

Remark 3.1 Theorem 3.2 can be rephrased as follows: $\bar{\mathbf{x}}$ is a weakly (respectively, properly) efficient solution of $(A-\mathrm{OWC}_{\mathcal{C}})$ if, and only if, there exists $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ (respectively, $\lambda \in \operatorname{int} \mathbb{R}^n_+$) and $\underline{\bar{\mathbb{Q}}} \in \mathcal{C}^n$ such that $(\bar{\mathbf{x}}, \underline{\bar{\mathbb{Q}}})$ is a saddle point of the function $\Xi_\lambda(\mathbf{x}, \underline{\mathbb{Q}})$ (see also (20)).

Fix $\bar{\mathbf{x}} \in \mathcal{X}$. We observe that, by Assumption 3.1.**(A1)**, the random variable

$$\bar{\mathbf{X}} = \varphi(\bar{\mathbf{x}}, \cdot)$$

belongs to L^∞_n. In the next result, we characterize the elements $\underline{\bar{\mathbb{Q}}} \in \mathcal{C}^n$ maximizing the function $\Xi_\lambda(\bar{\mathbf{x}}, \cdot)$ in terms of the subdifferential of the scalar map

$$\Phi \equiv \Phi_{\mathcal{C},A} \colon L^\infty_n \to \mathbb{R}$$

defined, for any $(\lambda = (\lambda_1, \dots, \lambda_n)^\top \in \mathbb{R}^n_+$ and) $\mathbf{Y} = (Y_1, \dots, Y_n)^\top \in L^\infty_n$, as

$$\begin{aligned} \Phi(\mathbf{Y}) &\coloneqq \max_{\underline{\mathbb{Q}} \in \mathcal{C}^n} \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\mathbb{Q}^{(1)}}[\mathbf{Y}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\mathbb{Q}^{(n)}}[\mathbf{Y}], A_n \rangle \end{bmatrix} \\ &\equiv \max_{\underline{\mathbb{Q}} \in \mathcal{C}^n} \sum_{i=1}^n \lambda_i \sum_{j=1}^n a_{i,j}\, \mathbb{E}_{\mathbb{Q}^{(i)}_j}[Y_j], \end{aligned} \tag{22}$$

where

$$\underline{\mathbb{Q}} = \left(\mathbb{Q}^{(1)}, \dots, \mathbb{Q}^{(n)}\right) \in \mathcal{C}^n, \quad \mathbb{Q}^{(i)} = \left(\mathbb{Q}^{(i)}_1, \dots, \mathbb{Q}^{(i)}_n\right) \in \mathcal{C}$$

and

$$\mathbb{E}_{\mathbb{Q}^{(i)}}[\mathbf{Y}] \coloneqq \begin{bmatrix} \mathbb{E}_{\mathbb{Q}^{(i)}_1}[Y_1] \\ \vdots \\ \mathbb{E}_{\mathbb{Q}^{(i)}_n}[Y_n] \end{bmatrix}$$

for all $i = 1, \dots, n$ (compare (22) with (20)).

Recall that, for any $\mathbf{X} \in L^\infty_n$, the subdifferential of Φ at $\mathbf{X}$ is defined by

$$\partial\Phi(\mathbf{X}) \coloneqq \Big\{ \ell \in \mathrm{ba}_n(\mathbb{P}) : \forall\, \mathbf{Y} \in L^\infty_n,\ \Phi(\mathbf{Y}) \geq \Phi(\mathbf{X}) + \ell(\mathbf{Y} - \mathbf{X}) \Big\}. \tag{23}$$

For any $\lambda \in \mathbb{R}^n_+$ and $\underline{\mathbb{Q}} = \left(\mathbb{Q}^{(1)}, \dots, \mathbb{Q}^{(n)}\right) \in \mathcal{C}^n$, we can associate the element $\ell \equiv \ell_{\lambda, \underline{\mathbb{Q}}, A}$ of $\mathrm{ba}_n(\mathbb{P})$ given by

$$\ell(\mathbf{Y}) \coloneqq \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\mathbb{Q}^{(1)}}[\mathbf{Y}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\mathbb{Q}^{(n)}}[\mathbf{Y}], A_n \rangle \end{bmatrix}, \quad \mathbf{Y} \in L^\infty_n \tag{24}$$

(see (22)). Accordingly, and with a slight abuse of notation, given $\lambda \in \mathbb{R}^n_+$, $\underline{\mathbb{Q}} \in \mathcal{C}^n$, and $\mathbf{X} \in L^\infty_n$, we will write

$$(\lambda, \underline{\mathbb{Q}}) \in \partial\Phi(\mathbf{X})$$

to indicate that the functional ℓ defined in (24) belongs to the set $\partial\Phi(\mathbf{X})$.

Proposition 3.1 *Let $\lambda \in \mathbb{R}^n_+$, $\bar{\mathbf{x}} \in \mathcal{X}$, and $\underline{\bar{\mathbb{Q}}} = \left(\bar{\mathbb{Q}}^{(1)}, \ldots, \bar{\mathbb{Q}}^{(n)}\right) \in \mathcal{C}^n$. Then, the following two conditions are equivalent (see* (20) *and* (23)*).*

(a) $\underline{\bar{\mathbb{Q}}} \in \operatorname{argmax}_{\underline{\mathbb{Q}} \in \mathcal{C}^n} \varXi_\lambda(\bar{\mathbf{x}}, \underline{\mathbb{Q}})$*;*

(b) $(\lambda, \underline{\bar{\mathbb{Q}}}) \in \partial \varPhi(\bar{\mathbf{X}})$.

Proof By (20) and (22),

$$\underline{\bar{\mathbb{Q}}} \in \operatorname{argmax}_{\underline{\mathbb{Q}} \in \mathcal{C}^n} \varXi_\lambda(\bar{\mathbf{x}}, \underline{\mathbb{Q}}) \quad \Longleftrightarrow \quad \varPhi(\bar{\mathbf{X}}) = \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\bar{\mathbf{X}}], A_n \rangle \end{bmatrix}.$$

Hence, assume first that condition **(a)** holds, i.e.,

$$\varPhi(\bar{\mathbf{X}}) = \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\bar{\mathbf{X}}], A_n \rangle \end{bmatrix}. \tag{25}$$

Then, for any $\mathbf{Y} \in L^\infty_n$, we have

$$\begin{aligned} \varPhi(\mathbf{Y}) &\geq \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\mathbf{Y}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\mathbf{Y}], A_n \rangle \end{bmatrix} \\ &\equiv \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\bar{\mathbf{X}}], A_n \rangle \end{bmatrix} + \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\mathbf{Y} - \bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\mathbf{Y} - \bar{\mathbf{X}}], A_n \rangle \end{bmatrix} \\ &= \varPhi(\bar{\mathbf{X}}) + \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\mathbf{Y} - \bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\mathbf{Y} - \bar{\mathbf{X}}], A_n \rangle \end{bmatrix}, \end{aligned}$$

which shows that $(\lambda, \underline{\bar{\mathbb{Q}}}) \in \partial \varPhi(\bar{\mathbf{X}})$, that is, condition **(b)** holds (refer also to (23)).

Conversely, assume **(b)** holds, i.e., $(\lambda, \underline{\bar{\mathbb{Q}}}) \in \partial \varPhi(\bar{\mathbf{X}})$. Then, taking $\mathbf{Y} = 0$ in the subdifferential inequality (see (23)), we obtain

$$0 = \varPhi(0) \geq \varPhi(\bar{\mathbf{X}}) + \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[-\bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[-\bar{\mathbf{X}}], A_n \rangle \end{bmatrix},$$

which implies

$$\varPhi(\bar{\mathbf{X}}) \leq \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\bar{\mathbf{X}}], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\bar{\mathbf{X}}], A_n \rangle \end{bmatrix}.$$

Since the reverse inequality is always satisfied by the definition of Φ as a supremum (see (22)), we conclude that equality must follow, i.e., (25) is verified. This yields that condition **(a)** holds. □

Starting from Theorem 3.2 and Proposition 3.1, we can derive a necessary and sufficient condition for optimality of the first order.

Proposition 3.2 Under Assumption 3.1, a point $\bar{\mathbf{x}} \in \mathcal{X}$ is a weakly (respectively, properly) efficient solution for problem $(A-\mathrm{OWC}_{\mathcal{C}})$ if, and only if, there exist $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ (respectively, $\lambda \in \operatorname{int} \mathbb{R}^n_+$) and a probability distribution $\underline{\bar{\mathbb{Q}}} = \left(\bar{\mathbb{Q}}^{(1)}, \ldots, \bar{\mathbb{Q}}^{(n)}\right) \in \mathcal{C}^n$ such that the following two conditions hold:

$$(\lambda, \underline{\bar{\mathbb{Q}}}) \in \partial\Phi(\bar{\mathbf{X}}), \tag{26}$$

$$0 \in \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\partial\varphi_{1,\omega}(\bar{\mathbf{x}})], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\partial\varphi_{n,\omega}(\bar{\mathbf{x}})], A_n \rangle \end{bmatrix} + N_{\mathcal{X}}(\bar{\mathbf{x}}). \tag{27}$$

Here, for any $i = 1, \ldots, n$ and ($\mathbb{P}$-almost every) $\omega \in \Omega$, $\partial\varphi_{i,\omega}(\bar{\mathbf{x}})$ denotes the subdifferential of $\varphi_i(\cdot, \omega)$ at $\bar{\mathbf{x}}$ and $N_{\mathcal{X}}(\bar{\mathbf{x}})$ denotes the normal cone to the set $\mathcal{X}$ at the point $\bar{\mathbf{x}} \in \mathcal{X}$.

Proof According to Theorem 3.2, $\bar{\mathbf{x}}$ is a weakly efficient solution of $(A-\mathrm{OWC}_{\mathcal{C}})$ if, and only if, there exist $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$ and $\underline{\bar{\mathbb{Q}}} \in \mathcal{C}^n$ such that the pair $(\bar{\mathbf{x}}, \underline{\bar{\mathbb{Q}}})$ is a saddle point of Ξ_λ: that is,

$$\textbf{(i)}\ \bar{\mathbf{x}} \in \operatorname{argmin}_{\mathbf{x}\in\mathcal{X}} \Xi_\lambda(\mathbf{x}, \underline{\bar{\mathbb{Q}}}) \quad \text{and} \quad \textbf{(ii)}\ \underline{\bar{\mathbb{Q}}} \in \operatorname{argmax}_{\underline{\mathbb{Q}}\in\mathcal{C}^n} \Xi_\lambda(\bar{\mathbf{x}}, \underline{\mathbb{Q}}) \tag{28}$$

(see (20)). Condition **(i)** simply state that $\bar{\mathbf{x}}$ is an optimal solution of the problem

$$\min_{\mathbf{x}\in\mathcal{X}} \Xi_\lambda(\mathbf{x}, \underline{\bar{\mathbb{Q}}}).$$

Since, by assumptions, $\Xi_\lambda(\cdot, \underline{\bar{\mathbb{Q}}})$ is convex and continuous at $\bar{\mathbf{x}}$, by standard optimality conditions $\bar{\mathbf{x}}$ is a minimizer of $\Xi_\lambda(\cdot, \underline{\bar{\mathbb{Q}}})$ over $\mathcal{X}$ if, and only if,

$$0 \in \partial\Xi_{\lambda,\underline{\bar{\mathbb{Q}}}}(\bar{\mathbf{x}}) + N_{\mathcal{X}}(\bar{\mathbf{x}}), \tag{29}$$

where $\partial\Xi_{\lambda,\underline{\bar{\mathbb{Q}}}}(\bar{\mathbf{x}})$ represents the subdifferential of $\Xi_\lambda(\cdot, \underline{\bar{\mathbb{Q}}})$ at $\bar{\mathbf{x}}$.

As established in [25, page 176], the subdifferential operator commutes with expectation in this setting. Together with the classical result that—under our standing regularity assumptions, namely convexity and continuity (as discussed just below equation (18))—the subdifferential of a finite sum coincides with the sum of the individual subdifferentials, this yields the following identity:

$$\partial\Xi_{\lambda,\underline{\bar{\mathbb{Q}}}}(\bar{\mathbf{x}}) = \lambda \cdot \begin{bmatrix} \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(1)}}[\partial\varphi_{1,\omega}(\bar{\mathbf{x}})], A_1 \rangle \\ \vdots \\ \langle \mathbb{E}_{\bar{\mathbb{Q}}^{(n)}}[\partial\varphi_{n,\omega}(\bar{\mathbf{x}})], A_n \rangle \end{bmatrix}$$

(see also [27] and references therein). This shows the equivalence between the inclusion (27) and condition **(i)** in (28).

The equivalence between (26) and condition **(ii)** in (28) has already been demonstrated in Proposition 3.1. This completes the proof. □

4 Connection with vector-valued risk measures

In this section, we investigate the link between the objective-wise worst-case problem ($A-\mathrm{OWC}_C$) introduced in Section 3 and the minimization of a vector-valued risk measure. We demonstrate that these two formulations are equivalent, thereby allowing the application of vector optimization techniques to minimize composite risk functions.

4.1 Definitions and properties

In the following, we introduce the notion of an $\mathbb{R}^n$-valued, normalized, and coherent risk measure. Our analysis is restricted to random vectors in $\mathcal{Y} = L_n^\infty$, whose elements represent the profit-and-loss outcomes (or returns) of financial positions, and whose riskiness is to be assessed via a vector-valued risk measure, denoted by $\boldsymbol{\varrho}$. Any element $\mathbf{X} \in L_n^\infty$ (i.e., a risky portfolio) consists of n components $X_1, \dots, X_n$, each belonging to L^∞.

As discussed in Section 2, we assume that $K \subseteq \mathbb{R}^n$ is the pointed and polyhedral cone of the form $K = A^{-1}\big(\mathbb{R}^n_+\big)$, where A is the nonsingular $n \times n$ matrix introduced in (6), and we additionally require that K contains $\mathbb{R}^n_+$. The partial ordering $\leq_K$ on $\mathbb{R}^n$, induced by K, naturally extends to L_n^∞ by:

$$0 \leq_K \mathbf{X} \quad \Longleftrightarrow \quad \mathbf{X} \in K \ (\mathbb{P}\text{-a.s.}).$$

Portfolios in L_n^∞ are thus (partially) ordered according to this rule.

An $\mathbb{R}^n$-valued risk measure is a map assigning to each $\mathbf{X} \in L_n^\infty$ an n-dimensional real vector, where the codomain $\mathbb{R}^n$ is equipped with the partial ordering $\leq_K$. The remainder of this section develops a coherent theory of such vector-valued risk measures within this cone-based framework.

Definition 4.1 (Vector-valued risk measure) A function

$$\boldsymbol{\varrho}\colon L_n^\infty \to \mathbb{R}^n$$

is called *vector-valued risk measure* with respect to K if the following two conditions hold:

(CA) for any $\mathbf{X} \in L_n^\infty$ and $\mathbf{u} \in \mathbb{R}^n$,

$$\boldsymbol{\varrho}(\mathbf{X} + \mathbf{u}) = \boldsymbol{\varrho}(\mathbf{X}) - \mathbf{u};$$

(M) for any $\mathbf{X}, \mathbf{Y} \in L_n^\infty$ with $\mathbf{Y} \leq_K \mathbf{X}$ $\mathbb{P}$-a.s.,

$$\boldsymbol{\varrho}(\mathbf{X}) \leq_K \boldsymbol{\varrho}(\mathbf{Y}).$$

$\boldsymbol{\varrho}$ is called *normalized* if

(N) $\boldsymbol{\varrho}(0) = 0$.

It is called *homogeneous* whenever

(H) for any $t \in]0, \infty[$ and $\mathbf{X} \in L_n^\infty$,

$$\boldsymbol{\varrho}(t\mathbf{X}) = t\,\boldsymbol{\varrho}(\mathbf{X}),$$

and *coherent* if, additionally,

(SA) for any $\mathbf{X}, \mathbf{Y} \in L_n^\infty$,

$$\boldsymbol{\varrho}(\mathbf{X} + \mathbf{Y}) \leq_K \boldsymbol{\varrho}(\mathbf{X}) + \boldsymbol{\varrho}(\mathbf{Y}).$$

Finally, $\boldsymbol{\varrho}$ is *convex* if

(CX) for any $\zeta \in]0, 1[$ and $\mathbf{X}, \mathbf{Y} \in L_n^\infty$,

$$\boldsymbol{\varrho}\big(\zeta\mathbf{X} + (1-\zeta)\mathbf{Y}\big) \leq_K \zeta\,\boldsymbol{\varrho}(\mathbf{X}) + (1-\zeta)\,\boldsymbol{\varrho}(\mathbf{Y}).$$

These axioms were first introduced for the univariate case ($n = 1$) in [2], then extended to the vector case in [21], to general ordered spaces in [7] and to the set-valued framework in [19]. Observe, in particular, that a vector-valued coherent risk measure is inherently convex.

Now, given any vector-valued and normalized risk measure $\boldsymbol{\varrho}$ with respect to K, we define its *acceptance set* as follows:

$$\mathscr{A}_{\boldsymbol{\varrho}} := \Big\{\mathbf{Y} \in L_n^\infty : \boldsymbol{\varrho}(\mathbf{Y}) \leq_K 0\Big\}. \tag{30}$$

Equivalently, a position $\mathbf{Y} \in L_n^\infty$ is *acceptable*, i.e., $\mathbf{Y} \in \mathscr{A}_{\boldsymbol{\varrho}}$, if, and only if,

$$-A \cdot \boldsymbol{\varrho}(\mathbf{Y}) \in \mathbb{R}_+^n,$$

in accordance with (6).

By conditions **(M)** and **(N)**, we have

$$L_{n,+}^\infty \subseteq \Big\{\mathbf{Y} \in L_n^\infty : \mathbf{Y} \in K\ (\mathbb{P}\text{-a.s.})\Big\} \subseteq \mathscr{A}_{\boldsymbol{\varrho}}.$$

If $\boldsymbol{\varrho}$ is homogeneous, then $\mathscr{A}_{\boldsymbol{\varrho}}$ forms a cone in L_n^∞; if $\boldsymbol{\varrho}$ fulfills the condition **(SA)**, then $\mathscr{A}_{\boldsymbol{\varrho}}$ is also convex. In particular, if $\boldsymbol{\varrho}$ is coherent, then $\mathscr{A}_{\boldsymbol{\varrho}}$ is a convex cone in L_n^∞ that contains the zero vector.

It deserves mention that all these properties were already established in the univariate case, formulated in a similar manner. For more comprehensive insights, please refer to [9] and [14, Chapter 4].

Further to this, we say that a vector-valued, normalized, and coherent risk measure $\boldsymbol{\varrho}$ with respect to K satisfies the *Fatou property* if, and only if, for every bounded sequence $\big(\mathbf{X}^{(m)}\big)_{m\in\mathbb{N}}$ converging $\mathbb{P}$-almost surely to some $\mathbf{X}$, it holds that

$$\boldsymbol{\varrho}(\mathbf{X}) \leq_K \liminf_{m\to\infty} \boldsymbol{\varrho}\big(\mathbf{X}^{(m)}\big).$$

Similar to the results presented in [14, Theorem 4.31], [7, Section 9], and [21, Theorems 4.1 and 4.2], it follows that, if $\boldsymbol{\varrho}$ is a vector-valued, normalized, and coherent risk measure with respect to K satisfying the Fatou property, then its acceptance set $\mathscr{A}_{\boldsymbol{\varrho}}$ is a weak*-closed convex cone in L_n^∞ containing the zero vector. Furthermore, by applying the classical bipolar theorem—adapted to the conic order induced by K—the cone $\mathscr{A}_{\boldsymbol{\varrho}}$ admits the following dual representation:

$$\mathscr{A}_{\boldsymbol{\varrho}} = \bigcap_{\mu \in \mathscr{A}_{\boldsymbol{\varrho}}^*} \Big\{ \mathbf{Y} \in L_n^\infty : \mathbb{E}_\mu[\mathbf{Y}] \geq 0 \Big\}, \tag{31}$$

where the set of dual elements is given by

$$\mathscr{A}_{\boldsymbol{\varrho}}^* := \big(\mathscr{A}_{\boldsymbol{\varrho}}\big)^* = \bigcap_{\mathbf{Y} \in \mathscr{A}_{\boldsymbol{\varrho}}} \Big\{ \mu \in \mathrm{ba}_n(\mathbb{P}) : \mathbb{E}_\mu[\mathbf{Y}] \geq 0 \Big\}. \tag{32}$$

Here, for any $\mathbf{X} = (X_1, \ldots, X_n)^\top \in L_n^\infty$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathrm{ba}_n(\mathbb{P})$,

$$\mathbb{E}_\mu[\mathbf{X}] = \sum_{j=1}^n \mathbb{E}_{\mu_j}[X_j], \qquad \mathbb{E}_{\mu_j}[X_j] = \int_\Omega X_j \, d\mu_j$$

for each $j = 1, \ldots, n$ (see Subsection 2.2).

4.2 A dual characterization

In this subsection, we derive a representation formula for vector-valued, normalized, and coherent risk measures $\boldsymbol{\varrho}$ with respect to the polyhedral cone K, assuming the Fatou property. This result extends the classical scalar case ($n = 1$) to a fully vectorial setting, providing a structure that highlights the link between marginal duality and global coherence, and offering a flexible framework for modeling multivariate risks under general conic monotonicity.

The characterization is constructed in a coordinatewise manner: each component of the risk measure is associated with a suitable family $\mathscr{P}_i$ of dual probability measures and a prescribed direction of evaluation. Thereafter, these directional elements are aggregated through a linear transformation, yielding the overall risk vector. The aforementioned procedure naturally relies on the matrix A spanning the cone K, as one might expect.

Before turning to this dual decomposition, it is worth emphasizing that the transformed map

$$\tilde{\boldsymbol{\varrho}} = (\tilde{\varrho}_1, \ldots, \tilde{\varrho}_n)^\top \colon L_n^\infty \to \mathbb{R}^n$$

defined, for any $\mathbf{Z} \in L_n^\infty$, as

$$\tilde{\boldsymbol{\varrho}}(\mathbf{Z}) := A\, \boldsymbol{\varrho}\big(A^{-1}\mathbf{Z}\big), \tag{33}$$

yields a vector-valued, normalized, and coherent risk measure with respect to the standard orthant $\mathbb{R}_+^n$, inheriting all constitutive properties of $\boldsymbol{\varrho}$, including the Fatou property. Conversely, given any vector-valued risk measure $\tilde{\boldsymbol{\varrho}}$ with respect to $\mathbb{R}^n_+$, the function

$$A^{-1}\,\tilde{\boldsymbol{\varrho}}(A\mathbf{X}), \quad \mathbf{X} \in L^\infty_n$$

defines a vector-valued risk measure with respect to K preserving analogous structural feature. These invariance properties, appertaining to the bijective transformations herein set forth, are as intuitive as they are readily verifiable.

We now introduce a preparatory lemma, which elucidates the interplay between the componentwise dual measures and the conic geometry governing the aggregation. This statement furnishes the technical foundation for the subsequent theorem and aligns with the formulations discussed in [7, Section 9], especially Example 9.1 therein.

Lemma 4.1 *Let $\boldsymbol{\varrho}\colon L^\infty_n \to \mathbb{R}^n$ be a vector-valued, normalized, and coherent risk measure with respect to K satisfying the Fatou property. For any $i = 1,\dots,n$, also let*

$$\mathscr{P}_i \equiv \mathscr{P}_{i,\boldsymbol{\varrho}}(\mathbb{P}) \coloneqq \Big\{\mathbb{Q}_i \in \mathcal{M}_{1,f}(\mathbb{P}) : \big(0,\dots,0,\mathbb{Q}_i,0,\dots,0\big) \in \mathscr{A}^*_{\tilde{\boldsymbol{\varrho}}}\Big\}, \tag{34}$$

where $\tilde{\boldsymbol{\varrho}}$ is the vector-valued risk measure with respect to $\mathbb{R}^n_+$ introduced in (33). *Then, for any $\mathbf{X} \in L^\infty_n$, any $i = 1,\dots,n$, and any $\mathbb{Q}_i \in \mathscr{P}_i$,*

$$\mathbb{E}_{\mathbb{Q}_i}[\langle -\mathbf{X}, A_i\rangle] \le \tilde{\varrho}_i(A\mathbf{X}) = \langle \boldsymbol{\varrho}(\mathbf{X}), A_i\rangle. \tag{35}$$

Proof We begin the proof by noting that condition **(CA)** and Definition (30) allow $\boldsymbol{\varrho}$ to be reconstructed from its acceptance set $\mathscr{A}_{\boldsymbol{\varrho}}$ as follows:

$$\boldsymbol{\varrho}(\mathbf{X}) = K\!-\mathrm{GLB}\Big\{\mathbf{y} \in \mathbb{R}^n : \mathbf{X} + \mathbf{y} \in \mathscr{A}_{\boldsymbol{\varrho}}\Big\}, \quad \mathbf{X} \in L^\infty_n \tag{36}$$

(thus, the above K-infimum is actually a K-minimum). By definition, this corresponds to the $K-$LUB with reversed sign of the shifted set

$$-\Big\{\mathbf{y} \in \mathbb{R}^n : \mathbf{X} + \mathbf{y} \in \mathscr{A}_{\boldsymbol{\varrho}}\Big\}$$

for every $\mathbf{X} \in L^\infty_n$ (refer also to (5) in Subsection 2.1).

In particular, taking into account (31)—a corollary of the bipolar theorem—and (36), one obtains that for any $\mathbf{X} \in L^\infty_n$, if we define

$$\mathcal{Y}_{\boldsymbol{\varrho},\mathbf{X}} \coloneqq \left\{\mathbf{y} = (y_1,\dots,y_n)^\top \in \mathbb{R}^n : \forall\, \mu \in \mathscr{A}^*_{\boldsymbol{\varrho}},\, \mathbb{E}_\mu[-\mathbf{X}] \le \sum_{j=1}^n y_j\,\mu_j(\Omega)\right\}, \tag{37}$$

then

$$\boldsymbol{\varrho}(\mathbf{X}) = K\!-\mathrm{GLB}\ \mathcal{Y}_{\boldsymbol{\varrho},\mathbf{X}}, \tag{38}$$

and this infimum is in fact attained, i.e., $\boldsymbol{\varrho}(\mathbf{X}) \in \mathcal{Y}_{\boldsymbol{\varrho},\mathbf{X}}$. In other terms, by (38), the i-th component of $A\,\boldsymbol{\varrho}(\mathbf{X})$ coincides with the infimum (or rather, the minimum) of the projection of the set $A\mathcal{Y}_{\boldsymbol{\varrho},\mathbf{X}}$ onto the i-th coordinate; that is,

$$\langle \boldsymbol{\varrho}(\mathbf{X}), A_i \rangle = \inf_{\mathbf{y}\in \mathcal{Y}_{\boldsymbol{\varrho},\mathbf{X}}} \langle \mathbf{y}, A_i \rangle.$$

At this point, we easily achieve (35) by (37) and (38), where the roles of $\boldsymbol{\varrho}$ and K are played by $\tilde{\boldsymbol{\varrho}}$ and $\mathbb{R}^n_+$, respectively. □

The dual representation theorem, presented and proved below, asserts that, for any $\mathbf{X} \in L^\infty_n$ and $i = 1, \dots, n$, the right-hand side of inequality (35) indeed coincides with the supremum of its left-hand side as $\mathbb{Q}_i$ ranges over $\mathscr{P}_i$.

While the proof follows the univariate case in spirit, it also builds on results from the set-valued framework—particularly [21, Theorems 4.1 and 4.2]—here adapted to our vector-valued setting. We also refer to [1, Section 3].

Theorem 4.1 *Let $\boldsymbol{\varrho}\colon L^\infty_n \to \mathbb{R}^n$ be a vector-valued, normalized risk measure with respect to K. Then, the following two statements are equivalent.*

1. *$\boldsymbol{\varrho}$ is coherent and satisfies the Fatou property.*
2. *For any $\mathbf{X} \in L^\infty_n$,*

$$\boldsymbol{\varrho}(\mathbf{X}) = A^{-1} \begin{bmatrix} \sup_{\mathbb{Q}_1\in\mathscr{P}_1} \mathbb{E}_{\mathbb{Q}_1}[\langle -\mathbf{X}, A_1\rangle] \\ \vdots \\ \sup_{\mathbb{Q}_n\in\mathscr{P}_n} \mathbb{E}_{\mathbb{Q}_n}[\langle -\mathbf{X}, A_n\rangle] \end{bmatrix}, \tag{39}$$

and each supremum is actually attained, with $\mathscr{P}_i$ denoting the family of dual probability measures defined in (34)*.*

Proof The core task of the proof consists in verifying that condition **1** implies condition **2**. The converse follows directly from the definitions, noting in particular that each set $\mathscr{P}_i$ is convex and weak*-compact.

Assume, then, that condition **1** holds. In order to derive (39), we initially observe that, by (33),

$$\langle \boldsymbol{\varrho}(A^{-1}\mathbf{Z}), A_i \rangle = \tilde{\varrho}_i(\mathbf{Z}), \quad \mathbf{Z} \in L^\infty_n, \quad i = 1, \dots, n,$$

so that establishing (39) amounts to proving that, for any $\mathbf{Z} = (Z_1, \dots, Z_n)^\top \in L^\infty_n$ and each $i = 1, \dots, n$,

$$\tilde{\varrho}_i(\mathbf{Z}) = \sup_{\mathbb{Q}_i\in\mathscr{P}_i} \mathbb{E}_{\mathbb{Q}_i}[-Z_i]. \tag{40}$$

More precisely, in view of Lemma 4.1 and with particular reference to equation (35), it suffices to prove the inequality "$\le$" in (40).

To this end, we recognize that each marginal $\tilde{\varrho}_i$ of $\tilde{\boldsymbol{\varrho}}$ is a proper, convex, and lower semicontinuous functional on L^∞_n, with respect to the weak* topology; consequently, it identifies with its biconjugate by virtue of the classical Fenchel-Moreau theorem (see [31, Section 2.3]): that is,

$$\tilde{\varrho}_i(\mathbf{Z}) = \sup_{\substack{\mathbf{U}\in L^1_n \\ \tilde{\varrho}^*_i(\mathbf{U})<\infty}} \Big\{ \mathbb{E}_{\mathbb{P}}[\langle \mathbf{U}, \mathbf{Z}\rangle] - \tilde{\varrho}^*_i(\mathbf{U}) \Big\}, \quad \mathbf{Z} \in L^\infty_n, \tag{41}$$

where

$$\tilde{\varrho}_i^*(\mathbf{U}) = \sup_{\mathbf{Z}\in L_n^\infty}\Big\{\mathbb{E}_{\mathbb{P}}[\langle \mathbf{U},\mathbf{Z}\rangle] - \tilde{\varrho}_i(\mathbf{Z})\Big\}, \quad \mathbf{U}\in L_n^1. \tag{42}$$

Fix $i \in \{1,\dots,n\}$. We examine the effective domain of the supremum in (41), that is, the set of directions $\mathbf{U}\in L_n^1$ for which the conjugate $\tilde{\varrho}_i^*(\mathbf{U})$ is finite—and thus may contribute nontrivially to the representation. As a first step, note that, since $L_{n,+}^\infty \subseteq \mathcal{A}_{\tilde{\varrho}_i}$, it immediately follows from (42) that, for any $\mathbf{U}\in L_n^1$,

$$\tilde{\varrho}_i^*(\mathbf{U}) \geq \sup_{\substack{\mathbf{Z}\in L_n^\infty\\ \tilde{\varrho}_i(\mathbf{Z})\leq 0}} \mathbb{E}_{\mathbb{P}}[\langle \mathbf{U},\mathbf{Z}\rangle] \geq \sup_{\mathbf{Z}\in L_{n,+}^\infty} \mathbb{E}_{\mathbb{P}}[\langle \mathbf{U},\mathbf{Z}\rangle],$$

and this implies that $\tilde{\varrho}_i^*(\mathbf{U}) < \infty$ holds only if $\mathbf{U}\leq 0$ ($\mathbb{P}$-a.s.), namely,

$$\mathbf{U}\in L_{n,-}^1. \tag{43}$$

Moreover, for any $\mathbf{U} = (U_1,\dots,U_n)^\top \in L_{n,-}^1$, we have

$$\begin{aligned}\tilde{\varrho}_i^*(\mathbf{U}) &\geq \sup_{\mathbf{u}\in\mathbb{R}^n}\Big\{\langle \mathbb{E}_{\mathbb{P}}[\mathbf{U}],\mathbf{u}\rangle + u_i\Big\}\\ &\equiv \sup_{\mathbf{u}\in\mathbb{R}^n}\left\{\big(\mathbb{E}_{\mathbb{P}}[U_i]+1\big)u_i + \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \mathbb{E}_{\mathbb{P}}[U_j]u_j\right\},\end{aligned}$$

as a consequence of the analogue of conditions **(CA)** and **(N)** for $\tilde{\boldsymbol{\varrho}}$. The finiteness of $\tilde{\varrho}_i^*(\mathbf{U})$ requires that the coefficients of the linear form above vanish:

$$\mathbb{E}_{\mathbb{P}}[U_j] = \begin{cases} 0, & \text{if } i\neq j,\\ -1, & \text{if } i=j,\end{cases}$$

for all $j = 1,\dots,n$. In light of (43), this entails that

$$-\mathbf{U}\in\boldsymbol{\Theta}_i,$$

where

$$\begin{aligned}\boldsymbol{\Theta}_i := \Bigg\{\boldsymbol{\theta} = (\theta_1,\dots,\theta_n)^\top \in L_{n,+}^1 : \forall\, j\in\{1,\dots,n\}\setminus\{i\},\, \theta_j = 0 \ (\mathbb{P}\text{-a.s.}),\\ \theta_i = \frac{d\mathbb{Q}_i}{d\mathbb{P}} \text{ for some } \mathbb{Q}_i\equiv\mathbb{Q}_{\theta_i}\in\mathcal{M}_{1,f}(\mathbb{P})\Bigg\}.\end{aligned}$$

Therefore, by (41), for any $\mathbf{Z} = (Z_1,\dots,Z_n)^\top \in L_n^\infty$,

$$\tilde{\varrho}_i(\mathbf{Z}) = \sup_{\substack{\boldsymbol{\theta}\in\boldsymbol{\Theta}_i\\ \tilde{\varrho}_i^*(-\boldsymbol{\theta})<\infty}} \Big\{\mathbb{E}_{\mathbb{P}}[-\theta_i Z_i] - \tilde{\varrho}_i^*(-\boldsymbol{\theta})\Big\}. \tag{44}$$

Now, since by (42), for any $\boldsymbol{\theta}\in\boldsymbol{\Theta}_i$,

$$\tilde{\varrho}_i^*(-\boldsymbol{\theta}) \geq \sup_{\mathbf{Y}\in\mathcal{A}_{\tilde{\boldsymbol{\varrho}}}} \mathbb{E}_{\mathbb{P}}[-\theta_i Y_i],$$

and given that $\mathscr{A}_{\tilde{\boldsymbol{\varrho}}}$ is a cone, a necessary condition for $\tilde{\varrho}_i^*(-\boldsymbol{\theta}) < \infty$ is that

$$\mathbb{E}_{\mathbb{P}}[\langle \boldsymbol{\theta}, \mathbf{Y} \rangle] \equiv \mathbb{E}_{\mathbb{P}}[\theta_i Y_i] \geq 0, \quad \mathbf{Y} = (Y_1, \ldots, Y_n)^\top \in \mathscr{A}_{\tilde{\boldsymbol{\varrho}}}. \tag{45}$$

Recalling (34) along with (32) specialized to $\tilde{\boldsymbol{\varrho}}$, it follows from (45) that, for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}_i$ with $\tilde{\varrho}_i^*(-\boldsymbol{\theta}) < \infty$,

$$\theta_i = \frac{d\mathbb{Q}_i}{d\mathbb{P}} \quad \text{for some} \quad \mathbb{Q}_i \equiv \mathbb{Q}_{\theta_i} \in \mathscr{P}_i. \tag{46}$$

Hence, by combining this with (42) and (35), we deduce that, for such $\boldsymbol{\theta}$,

$$\tilde{\varrho}_i^*(-\boldsymbol{\theta}) = \sup_{\mathbf{Z} \in L_n^\infty} \Big\{ \mathbb{E}_{\mathbb{Q}_i}[-Z_i] - \tilde{\varrho}_i(\mathbf{Z}) \Big\} \leq 0,$$

which, by the normalization property of $\tilde{\varrho}_i$, yields

$$\tilde{\varrho}_i^*(-\boldsymbol{\theta}) = 0. \tag{47}$$

In conclusion, from (44), (47), and (46), we find

$$\tilde{\varrho}_i(\mathbf{Z}) = \sup_{\substack{\boldsymbol{\theta} \in \boldsymbol{\Theta}_i \\ \tilde{\varrho}_i^*(-\boldsymbol{\theta}) < \infty}} \mathbb{E}_{\mathbb{P}}[-\theta_i Z_i] = \sup_{\mathbb{Q}_i \in \mathscr{P}_i} \mathbb{E}_{\mathbb{Q}_i}[-Z_i]$$

for all $\mathbf{Z} = (Z_1, \ldots, Z_n)^\top \in L_n^\infty$, as claimed in (40). This completes the argument. $\square$

The vector on the right-hand side of (39) is obtained by applying scalar, normalized, and coherent risk measures with the Fatou property to the projections of $-\mathbf{X}$ along fixed directions A_i, for $i = 1, \ldots, n$. Consequently, any vector-valued, normalized, and coherent risk measure $\boldsymbol{\varrho} \colon L_n^\infty \to \mathbb{R}^n$ satisfying the Fatou property admits a representation as an n-dimensional vector formed via directional evaluations of $-\mathbf{X}$, followed by linear aggregation via A^{-1}. This highlights a directional decomposition of $\boldsymbol{\varrho}$, which generalizes the marginal representations typically associated with the orthant-based setting.

4.3 Composite risk-based vector optimization

In many applications, uncertain outcomes are not directly observed, but arise as a result of decisions taken within complex stochastic systems. Formally, we consider

$$\mathbf{X} = \psi(\mathbf{x}),$$

where $\mathbf{x}$ belongs to the nonempty, closed, and convex set $\mathcal{X} \subseteq \mathbb{R}^k$, and

$$\psi \colon \mathcal{X} \to L_n^\infty$$

is a measurable mapping.

This motivates the study of composite risk functions of the form $\boldsymbol{\varrho} \circ \psi$, and of the associated vector optimization problem:

$$\leq_K - \min_{\mathbf{x} \in \mathcal{X}} \boldsymbol{\varrho}\big(\psi(\mathbf{x})\big), \tag{48}$$

where $\boldsymbol{\varrho}\colon L_n^\infty \to \mathbb{R}^n$ is a vector-valued risk measure with respect to K. Problem (48) characterizes the set of K-efficient points of the composition $\boldsymbol{\varrho} \circ \psi$.

If $\boldsymbol{\varrho}$ is normalized, coherent, and satisfies the Fatou property, then, by Theorem 4.1, problem (48) is equivalent to the robust objective-wise minimization problem:

$$\leq_K - \min_{\mathbf{x} \in \mathcal{X}} A^{-1} \begin{bmatrix} \sup_{\mathbb{Q}_1 \in \mathscr{P}_1} \mathbb{E}_{\mathbb{Q}_1}[\langle -\psi(\mathbf{x}), A_1 \rangle] \\ \vdots \\ \sup_{\mathbb{Q}_n \in \mathscr{P}_n} \mathbb{E}_{\mathbb{Q}_n}[\langle -\psi(\mathbf{x}), A_n \rangle] \end{bmatrix}. \tag{49}$$

This reformulation highlights the structure of (49) as an objective-wise worst-case optimization problem, in the spirit of (3).

Remark 4.1 If ψ satisfies axioms **(A1)**, **(A2)**, and **(A3)** of Assumption 3.1, then problem (49) fully fits within the theoretical framework developed in the previous sections.

In particular, if ψ is $\mathbb{P}$-a.s. K-concave, and $\boldsymbol{\varrho}$ is coherent, then the composite mapping $\boldsymbol{\varrho} \circ \psi$ is K-convex.

Indeed, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^k$ and $\zeta \in \,]0,1[$, $\mathbb{P}$-a.s. K-concavity of ψ means that, $\mathbb{P}$-a.s.,

$$\zeta\psi(\mathbf{x}_1) + (1-\zeta)\psi(\mathbf{x}_2) \leq_K \psi\big(\zeta\mathbf{x}_1 + (1-\zeta)\mathbf{x}_2\big).$$

Then, using the monotonicity (**(M)**) and convexity (**(CX)**) of $\boldsymbol{\varrho}$, we obtain:

$$\begin{aligned} \boldsymbol{\varrho}\big(\psi\big(\zeta\mathbf{x}_1 + (1-\zeta)\mathbf{x}_2\big)\big) &\leq_K \boldsymbol{\varrho}\big(\zeta\psi(\mathbf{x}_1) + (1-\zeta)\psi(\mathbf{x}_2)\big) \\ &\leq_K \zeta\,\boldsymbol{\varrho}\big(\psi(\mathbf{x}_1)\big) + (1-\zeta)\,\boldsymbol{\varrho}\big(\psi(\mathbf{x}_2)\big), \end{aligned}$$

which proves the K-convexity of $\boldsymbol{\varrho} \circ \psi$.

To establish an even stronger connection with the preceding parts of this work, and to reinforce the financial interpretation of the analysis, we now consider the case in which the ordering cone K is specialized to the nonnegative orthant $\mathbb{R}^n_+$. This choice reflects the standard componentwise ordering used in multivariate risk assessment.

Let us consider a vector function $\psi = (\psi_1, \ldots, \psi_n)^\top$, where each component ψ_i is assumed to be concave and, for simplicity, differentiable for all $i = 1, \ldots, n$. Under this setting, the optimality conditions developed in the previous section can be recast for the problem of minimizing a composite vector-valued risk function over the set

$$\mathcal{X} := \left\{ \mathbf{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k : x_j \in [0,1] \text{ for } j = 1, \ldots, k,\ \sum_{j=1}^k x_j \leq 1 \right\}.$$

We recall the following characterization of the normal cone (see Theorem 6.46 in [26]): for any $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_k)^\top \in \mathcal{X}$,

$$N_{\mathcal{X}}(\bar{\mathbf{x}}) = \Big\{ \mathbf{z} \in \mathbb{R}^n : \mathbf{z} = \alpha \mathbf{1} + (v_1, \ldots, v_n)^\top,\ \alpha \geq 0,\ v_i \leq 0,\ v_i = 0 \ \text{if} \ \bar{x}_i > 0 \Big\}. \tag{50}$$

Condition (27) entails that $\bar{\mathbf{x}} \in \mathcal{X}$ is a weakly risk minimizing portfolio if, and only if, there exist a nonzero vector $\lambda = (\lambda_1, \ldots, \lambda_n)^\top \in \mathbb{R}^n_+$ and a probability distribution $\bar{\mathbb{Q}} = (\bar{\mathbb{Q}}_1, \ldots, \bar{\mathbb{Q}}_n) \in \mathcal{C}$ such that

$$y \coloneqq \sum_{i=1}^n \lambda_i \, \mathbb{E}_{\bar{\mathbb{Q}}_i} \frac{\partial \psi_i}{\partial x_{j_1}}(\bar{\mathbf{x}}) = \cdots = \sum_{i=1}^n \lambda_i \, \mathbb{E}_{\bar{\mathbb{Q}}_i} \frac{\partial \psi_i}{\partial x_{j_m}}(\bar{\mathbf{x}}) \tag{51}$$

for $j_1, \ldots, j_m \in I \coloneqq \{i = 1, \ldots, k : \bar{x}_i > 0\}$, and

$$\sum_{i=1}^n \lambda_i \, \mathbb{E}_{\bar{\mathbb{Q}}_i} \frac{\partial \psi_i}{\partial x_{k_s}}(\bar{\mathbf{x}}) \leq y \tag{52}$$

for $k_s \in I_1 \coloneqq \{i = 1, \ldots, k : \bar{x}_i = 0\}$. Condition (51) states that there exist probability distributions such that at the optimal solution $\bar{\mathbf{x}}$ the weighted expected marginal contribution to the portfolio aggregation is equal for each asset entering the optimal solution. According to condition (52) assets with a lower weighted expected marginal contribution to the portfolio aggregation will not enter the optimal solution.

Remark 4.2 If we consider minimization of a scalar composite risk function, then conditions (51) and (52) reduce to

$$y \coloneqq \mathbb{E}_{\bar{\mathbb{Q}}} \frac{\partial \psi}{\partial x_{j_1}}(\bar{\mathbf{x}}) = \cdots = \mathbb{E}_{\bar{\mathbb{Q}}} \frac{\partial \psi}{\partial x_{j_m}}(\bar{\mathbf{x}}) \tag{53}$$

for $j_1, \ldots, j_m \in I \coloneqq \{i = 1, \ldots, k : \bar{x}_i > 0\}$, and

$$\mathbb{E}_{\bar{\mathbb{Q}}} \frac{\partial \psi}{\partial x_{k_s}}(\bar{\mathbf{x}}) \leq y \tag{54}$$

for $k_s \in I_1 \coloneqq \{i = 1, \ldots, k : \bar{x}_i = 0\}$.

Now, we consider the case of a linear portfolio aggregator. Let

$$\varPi = (\pi_{i,j})_{1 \leq i \leq n, 1 \leq j \leq k}$$

be a real matrix with n rows and k columns. We also explicitly introduce

$$\mathbf{R} = (R_1, \ldots, R_k)^\top \in L^\infty_k,$$

which one may naturally think of, for instance, as representing asset-wise payoffs (a viewpoint that will be formalized in the next section). Given a decision vector $\mathbf{x} = (x_1, \ldots, x_k)^\top \in \mathcal{X}$, we define the portfolio mapping $\psi \equiv \psi_{\mathbf{R},\varPi}$ by

$$\psi(\mathbf{x}) \coloneqq \varPi\big(\mathbf{R} \odot \mathbf{x}\big),$$

where

$$\mathbf{R} \odot \mathbf{x} = (x_1 R_1, \ldots, x_k R_k)^\top \in L^\infty_k \tag{55}$$

denotes the Hadamard (componentwise) product, defined pointwise $\mathbb{P}$-a.s. between the random vector $\mathbf{R}$ and the deterministic vector $\mathbf{x}$. In this case, condition 27 entails that $\bar{\mathbf{x}} \in \mathcal{X}$ is a weakly risk minimizing portfolio if, and only if, there exist a nonzero vector $\lambda = (\lambda_1, \ldots, \lambda_n)^\top \in \mathbb{R}^n_+$ and a probability distribution $\bar{\mathbb{Q}} = (\bar{\mathbb{Q}}_1, \ldots, \bar{\mathbb{Q}}_n) \in \mathcal{C}$ such that

$$\sum_{i=1}^n \lambda_i \langle \Pi_i, \bar{\mu} \rangle \in N_X(\bar{\mathbf{x}}), \tag{56}$$

where Π_i denotes the i-th row of Π, and $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_k)$ with $\bar{\mu}_i = \mathbb{E}_{\bar{\mathbb{Q}}_i}[\psi_i(\bar{\mathbf{x}})]$.

Condition (56) entails that $\bar{\mathbf{x}} \in \mathcal{X}$ is a weakly risk minimizing portfolio if, and only if, there exist a nonzero vector $\lambda \in \mathbb{R}^n_+$ and probability distributions $\bar{\mathbb{Q}}_i \in \mathcal{C}_i$ such that

$$y \coloneqq \sum_{i=1}^n \lambda_i \pi_{i,j_1} \bar{\mu}_{j_1} = \cdots = \sum_{i=1}^n \lambda_i \pi_{i,j_m} \bar{\mu}_{j_m} \tag{57}$$

for $j_1, \ldots, j_m \in J \coloneqq \{i = 1, \ldots, n : \bar{x}_i > 0\}$, and

$$\sum_{i=1}^n \lambda_i \pi_{i,k_s} \bar{\mu}_{k_s} \leq y \tag{58}$$

for $k_s \in J_1 \coloneqq \{i = 1, \ldots, n : \bar{x}_i = 0\}$. Condition (57) states that, at the optimal solution $\bar{\mathbf{x}}$, the weighted expected contribution to the portfolio aggregation must be equal for each asset entering the optimal solution. According to condition (58) assets with a lower weighted expected contribution to the portfolio aggregation will not enter the optimal solution.

Finally, we consider the case of separate linear aggregation, i.e., when, $\forall s \neq i$, $\pi_{i,j} \neq 0 \ \Rightarrow \ \pi_{s,j} = 0$. In this case conditions (57) and (58) become

$$y \coloneqq \lambda_{j_1} \pi_{i,j_1} \bar{\mu}_{i,j_1} = \cdots = \lambda_{j_m} \pi_{i,j_m} \bar{\mu}_{i,j_m} \tag{59}$$

for i and j_s such that $\pi_{i,j_s} \neq 0$ and $\bar{x}_{j_s} > 0$, and

$$\lambda_{k_s} \pi_{i,k_s} \bar{\mu}_{i,k_s} \leq y \tag{60}$$

for i and k_s such that $\pi_{i,j_s} \neq 0$ and $\bar{x}_{k_s} = 0$. Condition (59) states the existence of probability distributions such that, at the optimal solution $\bar{\mathbf{x}}$, the weighted expected return of each asset entering the optimal solution is equal.

5 An illustrative example

Multi-criteria optimization offers a well-established framework for analyzing portfolio selection problems in the spirit of Markowitz. We consider a financial market with k risky assets whose random outcomes are denoted by $R_1, \ldots, R_k$. A portfolio is represented by a vector $\mathbf{x} \in \mathbb{R}^k$, where each component specifies the proportion of wealth allocated to the corresponding asset.

Following the approach introduced in Section 3, we formulate the associated minimization problem in a finite-dimensional setting. Specifically, we assume that the underlying probability space is discrete:

$$\Omega = \{\omega_1, \ldots, \omega_d\}$$

for some $d \in \mathbb{N}$. The reference probability measure $\mathbb{P}$ is fully characterized by the point masses

$$p_j := \mathbb{P}(\{\omega_j\}), \quad j = 1, \ldots, d.$$

The random vector $\mathbf{R} = (R_1, \ldots, R_k)^\top$ of asset returns takes on d possible realizations, one for each scenario ω_j, denoted by $\mathbf{z}^j$ ($j = 1, \ldots, d$). The vector $\mathbf{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k$ satisfies the no-short-selling constraint and belongs to the feasible set

$$\mathcal{X} := \left\{ \mathbf{x} = (x_1, \ldots, x_k)^\top \in \mathbb{R}^k : x_i \in [0,1] \text{ for } i = 1, \ldots, k, \ \sum_{i=1}^{k} x_i \leq 1 \right\}.$$

The (random) value of each asset at the end of the investment period is $x_i W_0 R_i$ ($i = 1, \ldots, k$), where W_0 denotes the initial wealth (i.e., the total amount invested at time zero). Accordingly, the terminal value of the portfolio is represented by the k-dimensional random vector

$$W_0 \, \mathbf{R} \odot \mathbf{x} = W_0 \begin{bmatrix} x_1 R_1 \\ \vdots \\ x_k R_k \end{bmatrix}$$

(see the definition in (55)).

In this context, we address the issue of aggregation—namely, the passage from a vector of asset-wise outcomes in $\mathbb{R}^k$ to a vector-valued risk measure in $\mathbb{R}^n$. In line with the theoretical framework previously introduced, we assume that the ordering cone K is the nonnegative orthant:

$$K = \mathbb{R}^n_+.$$

For further details on portfolio aggregation and the associated terminology, we refer the reader to [21].

Precisely, we consider here a portfolio aggregator $f \equiv f_\Pi : \mathbb{R}^k \to \mathbb{R}^n$ of the form

$$f(\mathbf{z}) = \Pi \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^k$$

where $\mathit{\Pi}$ is an $n \times k$ real matrix. In this setting, $\psi\colon \mathcal{X} \to \mathbb{R}^n$ is given by

$$\psi(\mathbf{x}) := f\big(\mathbf{R} \odot \mathbf{x}\big) = \mathit{\Pi}\big(\mathbf{R} \odot \mathbf{x}\big), \quad \mathbf{x} \in \mathcal{X}.$$

To determine the risk associated to our portfolio we use the vector-valued Conditional Value at Risk associated to the vector of significance levels $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \,]0,1[^n$, i.e., we use the composite risk measure defined, for any $\mathbf{x} \in \mathcal{X}$, as

$$\mathrm{CVaR}_{\boldsymbol{\alpha}}\big(\psi(\mathbf{x})\big) = \begin{bmatrix} \mathrm{CVaR}_{\alpha_1}\big(\psi_1(\mathbf{x})\big) \\ \vdots \\ \mathrm{CVaR}_{\alpha_n}\big(\psi_n(\mathbf{x})\big) \end{bmatrix}.$$

As is well known, given the uncertain outcome $X \in L^\infty_1$, the single-valued $\mathrm{CVaR}_\alpha(\cdot)$, $\alpha \in \,]0,1[$, can be written as

$$\mathrm{CVaR}_\alpha(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}_{\mathbb{Q}}[-X], \tag{61}$$

for $\mathcal{Q}_\alpha$ being a suitable set of probability measures. Similarly, the vector-valued $\mathrm{CVaR}_{\boldsymbol{\alpha}}(\cdot)$ can be written, for any $\mathbf{x} \in \mathcal{X}$, as

$$\mathrm{CVaR}_{\boldsymbol{\alpha}}\big(\psi(\mathbf{x})\big) = \sup_{\mathbb{Q} \in \mathcal{Q}_{\boldsymbol{\alpha}}} \mathbb{E}_{\mathbb{Q}}[-\psi(\mathbf{x})], \tag{62}$$

for $\mathcal{Q}_{\boldsymbol{\alpha}}$ being a suitable set of vectors of probability measures. More precisely,

$$\mathcal{Q}_{\boldsymbol{\alpha}} := \mathcal{Q}_{\alpha_1} \times \cdots \times \mathcal{Q}_{\alpha_n},$$

with

$$\mathcal{Q}_{\alpha_i} := \left\{ \mathbb{Q}_i \in \mathcal{M}_{1,f}(\mathbb{P}) : \frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}} \leq \frac{1}{\alpha_i} \right\}, \quad i = 1, \dots, n \tag{63}$$

(refer to Theorem 4.47 in [14]). In the finite-dimensional setting, equality (63) can be written as

$$\mathcal{Q}_{\alpha_i} = \left\{ \mathbf{q}^i = (q^i_1, \dots, q^i_d)^\top \in \mathbb{R}^d_+ : \frac{q^i_j}{p_j} \leq \frac{1}{\alpha_i},\ \sum_{j=1}^d q^i_j = 1 \right\}$$

(see point (ii) of Theorem 1 in [23]). Now, for any $i = 1, \dots, n$, define:

$$\mathit{\Gamma}_{\alpha_i} := \left\{ \boldsymbol{\gamma}^i = (\gamma^i_1, \dots, \gamma^i_d)^\top \in \mathbb{R}^d_+ : \gamma^i_j \leq p_j,\ \sum_{j=1}^d \gamma^i_j = \alpha_i \right\}.$$

Then, expression (62) can be written equivalently through

$$\begin{aligned}
\mathrm{CVaR}_{\alpha_i}\big(\psi_i(\mathbf{x})\big) &= \sup_{\mathbf{q}^i\in\mathbf{Q}_{\alpha_i}} \sum_{j=1}^{d} -\mathit{\Pi}_i\,\mathbf{z}^j(\mathbf{x})q^i_j \\
&\equiv \sup_{\mathbf{q}^i\in\mathbf{Q}_{\alpha_i}} \frac{1}{\alpha_i}\sum_{j=1}^{d} -\mathit{\Pi}_i\,\mathbf{z}^j(\mathbf{x})\big(\alpha_i q^i_j\big) \\
&= \sup_{\boldsymbol{\gamma}^i\in\mathit{\Gamma}_{\alpha_i}} \frac{1}{\alpha_i}\sum_{j=1}^{d} -\mathit{\Pi}_i\,\mathbf{z}^j(\mathbf{x})\gamma^i_j \\
&= \sup_{\boldsymbol{\gamma}^i\in\mathit{\Gamma}_{\alpha_i}} \frac{1}{\alpha_i}\langle -\mathit{\Pi}_i\big(\mathbf{R}\odot\mathbf{x}\big),\boldsymbol{\gamma}^i\rangle
\end{aligned}$$

$(i=1,\dots,n)$, where $\mathit{\Pi}_i$ denotes the i-th row of $\mathit{\Pi}$ and $\mathbf{z}^j(\mathbf{x})$ denote j-th state of $\mathbf{R}\odot\mathbf{x}$ $(j=1,\dots,d)$. In particular, for any $i=1,\dots,n$ and $\mathbf{x}\in\mathcal{X}$,

$$\psi_i(\mathbf{x}) = \mathit{\Pi}_i\big(\mathbf{R}\odot\mathbf{x}\big).$$

Note that $\mathrm{CVaR}_{\alpha_i}\big(\psi_i(\cdot)\big)$ is a convex function on the whole $\mathcal{X}$ $(i=1,\dots,n)$.

Our minimization problem is thus

$$\begin{aligned}
&\min_{\mathbf{x}\in\mathcal{X}} \mathrm{CVaR}_{\boldsymbol{\alpha}}\big(\psi(\mathbf{x})\big) \\
&= \min_{\mathbf{x}\in\mathcal{X}} \begin{bmatrix} \sup_{\boldsymbol{\gamma}^1\in\mathit{\Gamma}_{\alpha_1}} \frac{1}{\alpha_1}\langle -\mathit{\Pi}_1\big(\mathbf{R}\odot\mathbf{x}\big),\boldsymbol{\gamma}^1\rangle \\ \vdots \\ \sup_{\boldsymbol{\gamma}^n\in\mathit{\Gamma}_{\alpha_n}} \frac{1}{\alpha_n}\langle -\mathit{\Pi}_n\big(\mathbf{R}\odot\mathbf{x}\big),\boldsymbol{\gamma}^n\rangle \end{bmatrix},
\end{aligned}$$

where the minimum is component-wise (Pareto minimization). We observe that, for any fixed $\mathbf{x}\in\mathcal{X}$, each component of the objective function above represents a linear programming problem that can be solved using well-known tools. Hence, for any $\mathbf{x}\in\mathcal{X}$, we can find (possibly nonunique) $\bar{\boldsymbol{\gamma}}^i(\mathbf{x})\in\mathit{\Gamma}_{\alpha_i}$, $i=1,\dots,n$ such that

$$\bar{\boldsymbol{\gamma}}^i(\mathbf{x}) \in \mathrm{argmax}_{\boldsymbol{\gamma}^i\in\mathit{\Gamma}_{\alpha_i}} \frac{1}{\alpha_i}\langle -\mathit{\Pi}_i\big(\mathbf{R}\odot\mathbf{x}\big),\boldsymbol{\gamma}^i\rangle.$$

Now, weakly or properly efficient solutions can be obtained from Theorem 3.2 by applying linear scalarization techniques to the multi-objective function

$$\begin{bmatrix} \frac{1}{\alpha_1}\langle -\mathit{\Pi}_1\big(\mathbf{R}\odot\mathbf{x}\big),\bar{\boldsymbol{\gamma}}^1(\mathbf{x})\rangle \\ \vdots \\ \frac{1}{\alpha_n}\langle -\mathit{\Pi}_n\big(\mathbf{R}\odot\mathbf{x}\big),\bar{\boldsymbol{\gamma}}^n(\mathbf{x})\rangle \end{bmatrix}, \quad \mathbf{x}\in\mathcal{X}.$$

To verify the portfolio performances under the proposed risk measure, we use returns from assets selected across different industrial sectors (the descriptive statistics are provided in Table 1).

Although the following results can also be obtained with a more realistic example (e.g., in a not finite probability space), the basic idea can be effectively illustrated in a simplified manner. Let us consider four assets with nominal parameters, including expected returns, volatilities, minimum and maximum values described by the table below.

Table 1: Descriptive statistics, where E1-E4 denote four stock returns representing companies from different sectors: Enel, Volkswagen, Hewlett Packard, and Intesa San Paolo. Data period: 2019-12-02 to 2020-11-27. Number of observations: 252

	E1	E2	E3	E4
Mean	0.00085	−0.00082	0.00160	−0.00063
Standard Deviation	0.02440	0.03347	0.02299	0.02851
Minimum	−0.22123	−0.15259	−0.11959	−0.19581
Maximum	0.07251	0.16697	0.17630	0.09263

We set $f(\mathbf{z}) = \varPi\mathbf{z}$, $\mathbf{z} \in \mathbb{R}^4$, with $\varPi \in \mathcal{M}(2 \times 4)$ being

$$\varPi \equiv \begin{bmatrix} \varPi_1 \\ \hline \varPi_2 \end{bmatrix} = \begin{bmatrix} 1\ 1\ 0\ 0 \\ 0\ 0\ 1\ 1 \end{bmatrix} \tag{64}$$

and we set $\alpha_1 = \alpha_2 = 0.1$.

The resulting efficient frontier, obtained by solving numerically the problem above from the weighted-sum scalarization described in Section 3 for 21 specific portfolios (with $\lambda = 0.05h$, $h = 0, \ldots, 20$), is illustrated in Figure 1.

Figure 1 also illustrates the image set of the bi-objective function

$$\begin{bmatrix} \mathrm{CVaR}_{\alpha_1}\big(-\varPi_1(\mathbf{R}\odot\mathbf{x})\big) \\ \mathrm{CVaR}_{\alpha_2}\big(-\varPi_2(\mathbf{R}\odot\mathbf{x})\big) \end{bmatrix}$$

for 500 portfolios $\mathbf{x} \in \mathcal{X}$. Clearly, these are always located to the top right of the efficient frontier.

Notice that the efficient frontier and the image set depend on the specific problem data. Setting $\varPi \in \mathcal{M}(2 \times 4)$ equal to

$$\varPi = \begin{bmatrix} 1\ 0\ 2\ 1 \\ 0\ 1\ 1\ 3 \end{bmatrix} \tag{65}$$

and leaving $\alpha_1 = \alpha_2 = 0.1$, a different picture can be obtained; see Figure 2. We notice moreover that, in this case, the image set fails to be convex.

6 Conclusions

This work proposed a robust framework for vector optimization under uncertainty, built on a component-wise interpretation of worst-case analysis. The formulation is particularly suited to multi-objective decision problems in which

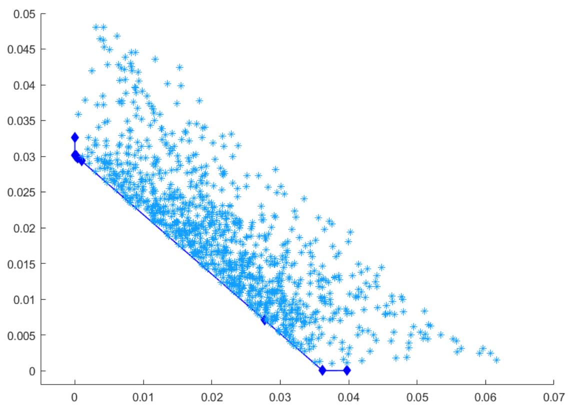

Fig. 1: Efficient frontier and image set with Π as in (64)

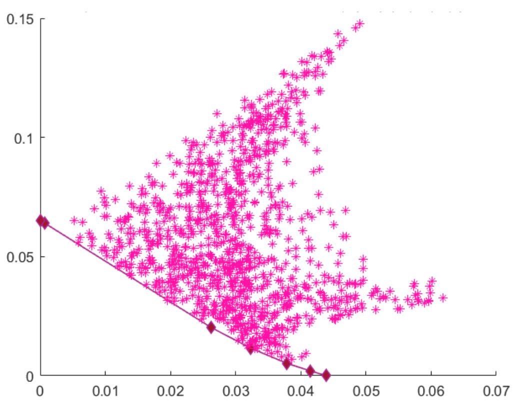

Fig. 2: Efficient frontier and image set with Π as in (65)

model parameters, such as random returns or costs, are only partially known and must be estimated.

In the first part, we extended the classical minimax formulation to the multi-objective setting by replacing each component of the objective with its worst-case counterpart. Under mild assumptions, this approach preserves convexity and remains compatible with efficient frontier concepts.

In the second part, we investigated coherent vector-valued risk measures and established a dual representation under the Fatou property. This result allows composite problems to be reformulated as robust optimization models involving vector expectations over sets of probability measures.

A central structural element is the ordering cone $K = A^{-1}(\mathbb{R}^n_+)$, which induces a partial order $\leq_K$ on $\mathbb{R}^n$ and supports the formulation of the vector optimization problem. A key assumption is that the matrix A is square and invertible, ensuring that K is full-dimensional and that the K-supremum reduces to a single vector.

When A is not invertible or not square, the supremum no longer corresponds to a unique vector but becomes a set of nondominated points. This shift requires a set-valued reformulation with alternative notions of supremum to maintain coherence. Although scalarization-based approaches (e.g., [10], [11]) and lattice-theoretic tools (e.g., [17], [18]) provide feasible treatments, they depart from the purely vector-valued setting considered here. An alternative perspective is offered by the duality-based framework in [19], which yields robust counterparts based on conic closures and set-valued risk epigraphs.

These approaches illustrate distinct ways to handle partial orders and non-singleton outcomes. While our analysis was restricted to the vectorial case induced by an invertible matrix, extending the framework to more general ordering structures remains an open and promising direction for future research.

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