Stochastic orderings for set-valued risk measures

Elisa Mastrogiacomo · Marco Tarsia

Abstract. This paper explores the portfolio aggregation problem within the framework of set-valued risk measures, with a specific emphasis on maximal correlation risk measures, as introduced in this work. We propose a novel stochastic ordering concept for random vectors and establish consistency properties for maximal correlation risk measures in this setting. Furthermore, we demonstrate convex-type consistency for a specific subclass of law-invariant convex set-valued risk measures, highlighting both their theoretical foundations and practical significance.

Key words. Law-invariant convex risk measure, portfolio aggregator, set-valued risk measure, multivariate stochastic ordering, set-valued upper expectation.

AMS 2000 subject classifications. 26E25, 47H04, 60E15, 62P05, 91B30, 91G70.

Introduction

The quantitative assessment of risk has long been central in actuarial science and statistical decision theory, where rigorous approaches to modelling uncertainty and capital adequacy were already firmly established; see, for instance, [26] and [19]. Its evolution into mathematical finance in the late 1990s, marked by the seminal contribution [2] and subsequently refined in [11, 10], introduced the coherent framework and its essential axioms.

Comprehensive surveys on scalar risk measures can be found in [16] and [18], where convexity is emphasized as a structural requirement rather than merely an axiom. This formalization captures the core principle whereby diversification mitigates risk. Additional insights into convex risk models are available in [29].

Risk quantification is found to play a key role in financial modelling by assessing investment risks and optimizing capital allocation. While traditionally applied in a univariate and static setting, where potential losses are quantified in monetary units and greater losses are indicative of higher risk, the study of multivariate portfolios risks entails greater complexity and interest. Recent research advancements suggest sophisticated alternatives to aggregating assets based simply on their cash equivalents. Notably, [6] describes risk measures as functions mapping into a partially ordered cone, a perspective further explored in [7].

In [27], vector-valued coherent risk measures are defined, and the aggregation issue is also studied. Precisely, given $d, n \in \mathbb{N}$ with $1 \le d \le n$, a transformation from an n-dimensional random portfolio vector X to a $\mathbb{R}^d$-valued coherent risk measure $\widetilde{R}$ is considered. This operation involves composing a function $f\colon \mathbb{R}^n \to \mathbb{R}^d$, satisfying appropriate conditions and referred to as an aggregator, with a complementary $\mathbb{R}^d$-valued risk measure R, i.e.,

$$\widetilde{R}(X) = R\big(f(X)\big).$$

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In the following, we shall refer to these functionals as composite risk measures, a terminology already adopted in several papers within the literature, including [34] and [35], among others.

Set-valued convex risk measures are defined in [21] as an extension of vector-valued coherent risk measures by omitting the homogeneity axiom (see also [20]). The paper also offers notable dual representation theorems framed in the context of convex analysis.

In risk management, stochastic orderings are employed to define desirable characteristics of risk measures. For instance, a risk-averse rational decision-maker would favor a risky position that is lower according to a given “reasonable” stochastic order $\leq_{\mathrm{st}}$. This is highly relevant to what are classified as convex orderings. The consistency, or monotonicity, of a risk measure with respect to a stochastic ordering can be viewed as a key property, furnishing a clearer understanding than the measure's convexity on its own. Additionally, this approach allows for capturing the joint risk of a portfolio by accounting for the variations in its components and their potential dependencies.

Although the literature has explored the consistency of risk measures, such analyses have been predominantly univariate. For a fuller review, see [4] alongside other significant sources.

In light of the preceding discussion, we extend the findings of [5] to this set-valued scenario, investigating the interplay between univariate risk measures applied to transformed portfolios and their multivariate counterparts.

Our contribution unfolds through the following procedure. We begin by examining composite set-valued risk measures, specifically identifying the conditions on the aggregator under which the axioms satisfied by the original risk measure are preserved in the composite formulation. Next, we explore the properties of the maximal correlation set-valued risk measure, conceptualized as the intersection of (the values of) set-valued risk measures, with its structure inherently determined by the axioms satisfied by its constituent measures. The intersection operation effectively identifies the maximal element with respect to a suitable order relation on an appropriately defined class of sets. Consequently, the characterization obtained in this framework holds a fortiori when the constituent risk measures themselves take the form of composite measures.

Building upon this foundation, we then introduce a notion of multivariate stochastic ordering, denoted by $\leq_{\mathcal{F}^{n,d}}$, for $\mathbb{R}^n$-valued random vectors. This order is defined in relation to broad function classes $\mathcal{F}^{n,d}$ mapping $\mathbb{R}^n$ to $\mathbb{R}^d$, utilizing the upper expectation operator as presented in [23] and [22], which bring essentially the same definition in our setting. Subsequently, we assess the consistency of specific set-valued maps Ψ_A, of the form

$$\Psi_A(X) = \bigcap_{\alpha \in A} \Psi_\alpha\big(g_\alpha(X)\big),$$

possibly interpretable as a type of a maximal correlation risk measure, with respect to the stochastic ordering $\leq_{\mathcal{F}^{n,d}}$. This analysis establishes conditions under which the following implication—exactly characterizing consistency—holds:

$$X \leq_{\mathcal{F}^{n,d}} Y \quad \Longrightarrow \quad \Psi_A(Y) \subseteq \Psi_A(X).$$

To the best of the authors' knowledge, the consistency properties of set-valued risk measures have yet to be explored in this generality.

In particular, we implement this framework for the class

$$\mathcal{F}^{n,d} = \mathcal{F}^{n,d}_{\prec}$$

which consists of functions exhibiting Schur-convexity with respect to a given cone. Originally introduced by Issai Schur in 1923 (see [37]), Schur-convex functions are fundamental in the study of ordered structures, as they preserve the majorization relation. This property has far-reaching implications across multiple domains, including optimization, economics, and statistics, as it naturally encodes notions of balance, dispersion, and fairness.

By generalizing the classical notion of Schur-convexity to vector-valued functions and building upon the results established in [33], our work provides a unified perspective that connects majorization theory with multivariate risk assessment. This broader framework enriches the conceptual landscape while enhancing the applicability of Schur-convex functions in contexts where multidimensional comparisons are essential. The proposed approach also uncovers new structural properties of set-valued risk measures and stochastic orders, offering a more robust mathematical foundation for analyzing dependencies in high-dimensional settings.

Lastly, we consider law-invariant convex set-valued risk measures that admit a dual representation. These measures play a fundamental role in risk evaluation by ensuring that risk assessments depend solely on the distribution of financial positions rather than their specific realizations. This property enhances their applicability in regulatory and decision-making contexts, while the dual characterization reveals deeper insight into their structural properties and reinforces their significance in practical risk management. In particular, we focus on the set-valued distortion risk measure, otherwise known as the weighted value at risk (see, e.g., [9] and [12]).

The paper is structured as follows. Section 1 introduces the necessary notations and definitions (Subsection 1.1), and explores the integration of the concepts of risk measure and portfolio aggregation (Subsection 1.2). Section 2 begins with an overview of stochastic orderings, followed by a comprehensive examination of the consistency under convex-type stochastic orderings of a maximal correlation risk measure (Subsection 2.1), detailing its relationship with the class of the Schur-convex functions (Subsection 2.2). Finally, in Section 3, we offer results on the consistency for law-invariant convex set-valued risk measures that enable a dual representation.

1 Set-valued risk measures and portfolio aggregators

The foundation of this first section is laid by discussing the fundamental properties of *composite* set-valued risk measures: namely, set-valued mappings obtained by composing a set-valued risk measure with a *portfolio aggregator* akin to those considered in [27]. We then propose an extension of the *maximal correlation risk measure*, adopting an approach similar to that in [5], within a set-valued framework analogous to the one outlined in [21]. In this formulation, risk values are characterized as the intersection of set-valued risk measures. In both cases, we investigate the preservation of the axioms of set-valued risk measures across the respective constituent risk measures.

Lastly, we refine this general setting by focusing on the specific case in which the maximal correlation risk measure is constructed from composite risk measures. In this connection, we wish to anticipate that we will revisit this topic in Section 2, where we assess its potential *consistency* with respect to certain stochastic orderings.

1.1 Notations and definitions

Building primarily on the methodology developed in [27] and [21] as previously delineated, we approach the problem by introducing several essential tools. First, we select three positive integers $m, d, n \in \mathbb{N}^* \doteq \mathbb{N} \setminus \{0\}$, with $m \leq d \leq n$, and their corresponding Euclidean spaces, which are regarded as being geometrically and topologically embedded in the canonical manner.

More precisely, we identify $\mathbb{R}^m$ with the subspace

$$\left\{ \tilde{u} = [\tilde{u}_1, \ldots, \tilde{u}_n]^\mathsf{T} \in \mathbb{R}^n \;\middle|\; \forall\, i = m+1, \ldots, n,\ \tilde{u}_i = 0 \right\}, \tag{1}$$

and, analogously, $\mathbb{R}^d$ with

$$\left\{ \tilde{v} = [\tilde{v}_1, \ldots, \tilde{v}_n]^\mathsf{T} \in \mathbb{R}^n \;\middle|\; \forall\, j = d+1, \ldots, n,\ \tilde{v}_j = 0 \right\},$$

thereby yielding the natural inclusions

$$\mathbb{R}^m \subseteq \mathbb{R}^d \subseteq \mathbb{R}^n.$$

In the context of addressing portfolio aggregation problems, it is customary to regard d and n as non-coinciding; that is,

$$d < n.$$

Within these spaces, we focus especially on two *closed*, *convex* and *pointed cones*

$$K \subseteq \mathbb{R}^d, \quad \widetilde{K} \subseteq \mathbb{R}^n$$

both containing the respective origin (and, in the special case where $n = d$, typically distinct). Recall that a cone in a Euclidean spaces is convex if, and only if, it is closed under the addition of its elements. Afterward, according to the general definition of a pointed cone (in a real topological vector space),

$$K \cap -K = \{0\}, \quad \widetilde{K} \cap -\widetilde{K} = \{0\}.$$

See, for example, [3] or [32] among the many possibilities.

For the sake of notational simplicity, we shall in the sequel employ the plain symbol 0 to signify, as context dictates, the zero vector as well, whether it be deterministic or random, and irrespective of its dimension or underlying probabilistic nature.

Remark 1.1. Under this setup, we will embrace different symbolic notations depending on whether we are dealing with K (in $\mathbb{R}^d$) or $\widetilde{K}$ (in $\mathbb{R}^n$), including for the measurable random vectors that will be introduced shortly. We believe that this distinction will greatly enhance the clarity and comprehensibility of the technical results presented in the upcoming sections.

The *vector partial orders* $\leq_K$ and $\leq_{\widetilde{K}}$, respectively *generated* or *induced by* K and $\widetilde{K}$, are described in the following way (see, e.g., [24]): for any $u, v \in \mathbb{R}^d$ and $x, y \in \mathbb{R}^n$,

$$u \leq_K v \quad \Longleftrightarrow \quad v - u \in K, \quad x \leq_{\widetilde{K}} y \quad \Longleftrightarrow \quad y - x \in \widetilde{K}.$$

These binary relations are reflexive and transitive as preorders, antisymmetric due to the assumed pointedness of K and $\widetilde{K}$, and compatible with the algebraic operations of $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively.

Next, we take into account a *m-dimensional vector subspace*

$$M \equiv M\big(K,\widetilde{K}\big) \subseteq \mathbb{R}^d$$

of $\mathbb{R}^d$ (potentially depending on K and $\widetilde{K}$) such that, in $\mathbb{R}^n$, the intersection of M with $\widetilde{K}$ reduces to the intersection of M with K (in $\mathbb{R}^d$). Herein, all arguments remain framed within the natural embedding mentioned above; specifically, a projection is applied in this context. In addition, the interior of $K \cap M \equiv \widetilde{K} \cap M$ in $\mathbb{R}^d$ is supposed to be nonempty. To symbolize both assumptions:

$$K \cap M = \widetilde{K} \cap M, \qquad \mathrm{int}(K \cap M) \equiv \mathrm{int}_{\mathbb{R}^d}(K \cap M) \neq \emptyset. \tag{2}$$

This intersection is, in turn, a closed, convex and pointed cone containing the origin.

In the spirit of [23], the linear subspace M is construed as the space of *eligible assets/portfolios*, namely those positions that a regulatory authority may deem admissible for initial deposits or reserve holdings. As not all assets are generally accepted for such purposes, it is customary to have

$$m < d.$$

We then introduce the family of *upper closed* subsets of M, relative to K, i.e.,

$$\mathbb{F}_M := \Big\{ D \subseteq M \;\Big|\; D = \mathrm{cl}\Big(D + \big(K \cap M\big)\Big)\Big\}, \tag{3}$$

as defined in [21]. Here, $\mathrm{cl}(\cdot)$ denotes the usual (topological) closure operator, while $+$ refers to the standard Minkowski addition. In particular,

$$\emptyset \in \mathbb{F}_M, \tag{4}$$

since $\emptyset = \emptyset + \big(K \cap M\big)$ by definition. Moreover, $\mathbb{F}_M$ is closed under arbitrary intersections.

Now, we consider a *complete* probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and, for any $\ell \in \mathbb{N}^*$, here below and henceforth, we let

$$L^\infty_\ell(\Omega) \equiv L^\infty\big((\Omega, \mathcal{F}, \mathbf{P}); \mathbb{R}^\ell\big)$$

represent the space of the $\mathbf{P}$-a.s. bounded, $\mathbb{R}^\ell$-valued, $\mathcal{F}$-measurable random variables defined on Ω (identified with their respective equivalence class under $\mathbf{P}$-a.s. equality). As a special case, the zero vector, like any other constant vector in $\mathbb{R}^\ell$, is treated as an element of $L^\infty_\ell(\Omega)$.

In financial terms, this space can be interpreted as modeling random vectors that represent uncertain outcomes, such as asset returns or cash flows, whose values are constrained within a bounded set. This ensures that the associated risks and returns are both measurable and restricted to a finite range, with respect to the underlying probability measure $\mathbf{P}$.

In full accordance with [21, Definition 2.1], we refer to a *set-valued risk measure associated with (M and) K* as a set-valued function

$$R\colon L^\infty_d(\Omega) \to \mathbb{F}_M$$

satisfying the following three properties (refer to (3) for the definition of $\mathbb{F}_M$).

(R-nor) — *normalization (w.r.t. K and M)*:

$$K \cap M \subseteq R(0), \qquad R(0) \cap -\operatorname{int}(K \cap M) = \emptyset.$$

(R-inv) — *translation invariance in M*: for any $U \in L^\infty_d(\Omega)$ and $u \in M$,

$$R(U + u) = R(U) + \{-u\}. \tag{5}$$

(R-mon) — *monotonicity (w.r.t. K)*: for any $U, V \in L^\infty_d(\Omega)$,

$$U \leq_K V \ (\mathbf{P}\text{-a.s.}) \quad \Longrightarrow \quad R(U) \subseteq R(V).$$

Concerning (5) and similar notations, we will use the following intuitive and widely recognized symbolism: for any $U \in L^\infty_d(\Omega)$ and $u \in M$,

$$R(U) - u \coloneqq R(U) + \{-u\}.$$

As a matter of course, the *effective domain* of $R(\cdot)$ will be identified by the set

$$\operatorname{dom}(R) \doteq \big\{U \in L^\infty_d(\Omega) \,\big|\, R(U) \neq \emptyset\big\}. \tag{6}$$

Remark 1.2. We draw attention to the fact that, for any $U \in L^\infty_d(\Omega)$, one actually has $R(U) \in \mathbb{F}_M$. Indeed, if $U \notin \operatorname{dom}(R)$, then $R(U) = \emptyset \in \mathbb{F}_M$, as follows from (6) and (4). Conversely, if $U \in \operatorname{dom}(R)$, then, simply by definition, $R(U) \neq \emptyset$ and $R(U) \in \mathbb{F}_M$.

Therefore, as a further consequence to be invoked in due course, based on (3), we derive

$$R(U) + \big(K \cap M\big) \subseteq R(U) \tag{7}$$

(every subset of any Euclidean space is contained in its own closure).

The axioms defining a set-valued risk measure represent essential financial principles for evaluating and managing risk effectively. Normalization establishes a consistent framework by requiring that every position in $K \cap M$ is considered acceptable (in the absence of additional risk), while explicitly preventing contradictions such as overlapping opposite directions within this intersection. Translation invariance in M indicates that adding a position from M shifts the risk set by the same amount. This reflects the idea that elements of M represent neutral adjustments that do not affect the underlying risk but merely translate its evaluation. Monotonicity conveys that a stronger financial position, as ordered by K, cannot result in a higher risk level. This property upholds the principle that risk measures must respect dominance and provide consistent evaluations of comparable portfolios.

We direct the reader to the first part of [21] for more details regarding the above definitions, including any relevant comparisons with earlier literature on this topic. In particular, among the various references, we find it relevant to mention [27, Definition 2.1], where the axiom of *coherence* is applied to the target set-valued risk measure from the outset, in contrast to our approach.

We point out that, by **(R-nor)**, since $0 \in K \cap M$, we obtain

$$0 \in R(0). \tag{8}$$

In particular, $R(0) \neq \emptyset$ and thus, according to (6), $0 \in \mathrm{dom}(R)$. Going further, for any $U \in L_d^\infty(\Omega)$,

$$U \in K \ (\mathbf{P}\text{-a.s.}) \quad \Longrightarrow \quad 0 \in R(U) \quad \Longrightarrow \quad U \in \mathrm{dom}(R), \tag{9}$$

through the application of **(R-mon)** and (8) (refer to (6) once again).

Remark 1.3. The adoption of the term *normalization*, with respect to **(R-nor)**, is reinforced by the observation that (8) conversely implies that $K \cap M \subseteq R(0)$, both of which are ultimately equivalent. Indeed, it suffices to apply (7), where $U = 0$ ($\mathbf{P}$-a.s.), and (8) to reach that $K \cap M \subseteq R(0)$.

We present two supplementary conditions, *stronger* then **(R-nor)** and **(R-mon)** respectively, which we subsequently comment on in some detail.

(R-nor$_\mathbf{s}$) — *strong normalization (w.r.t. K and M)*:

$$K \cap M = R(0). \tag{10}$$

(R-mon$_\mathbf{s}$) — *strong monotonicity*: for any $U, V \in L_d^\infty(\Omega)$,

$$V - U \in R(0) \ (\mathbf{P}\text{-a.s.}) \quad \Longrightarrow \quad R(U) \subseteq R(V). \tag{11}$$

Numerous normalization properties for a set-valued risk measure $R(\cdot)$ have been extensively examined in the literature. Among these, one notably prevalent and broadly adopted is illustrated below and warrants special attention: it can be interpreted as a form of subadditivity with respect to the origin, wherein $R(0)$ acts as a neutral element.

(R-nor*) — $R(0)$*-additive invariance*: $R(0) \notin \{\emptyset, M\}$ and, for any $U \in L_d^\infty(\Omega)$,

$$R(U) + R(0) = R(U). \tag{12}$$

Remark 1.4. We emphasize that **(R-nor*)** implies **(R-nor)** if the graph of $R(\cdot)$, defined by

$$\mathrm{graph}(R) \doteq \Big\{ (U,u) \in L_d^\infty(\Omega) \times M \ \Big| \ u \in R(U) \Big\},$$

turns out to be closed with respect to the product topology (i.e., if $R(\cdot)$ is *closed*)—specifically, the weak-$*$ topology $\sigma\big(L_d^\infty(\Omega), L_d^1(\Omega)\big)$ on $L_d^\infty(\Omega)$ and the subspace topology on $M \subseteq \mathbb{R}^d$. The key intuition is that the closedness of $\mathrm{graph}(R)$ ensures the stability of the inclusion defining $R(0)$ under limit operations, with weak closure playing a crucial role in preserving this consistency. For a deeper discussion, refer, for instance, to [21] and [13, Subsection 2.1].

In the case where (8) is fulfilled, as it is within our framework, (12) is clearly equivalent to the condition that, for any $U \in L_d^\infty(\Omega)$,

$$R(U) + R(0) \subseteq R(U) \tag{13}$$

(the reverse inclusion holds trivially, since any random vector can be decomposed as the sum of itself and the zero vector).

As a consequence, **(R-nor$_\mathbf{s}$)** proves to be a stronger axiom then (12) as well because, if (10) is assumed, then, for any $U \in L_d^\infty(\Omega)$, (13) is a straightforward corollary of (7) (Remark 1.2).

In a scenario of this kind, it can be inferred, specifically, that

$$R(0) + R(0) = R(0). \tag{14}$$

Furthermore, given that K is pointed, under $(R\textbf{-nor}_\mathbf{s})$ it is evident that

$$R(0) \cap -R(0) = \{0\}. \tag{15}$$

With regard to $(R\textbf{-mon}_\mathbf{s})$, its formulation directly depends on the origin (of $\mathbb{R}^d$) and only indirectly on the entirety of the cone K, via its influence on the map $R(\cdot)$. Additionally, if $(R\textbf{-mon}_\mathbf{s})$ holds, then, by (8), for any $U \in L^\infty_d(\Omega)$,

$$U \in R(0)\ (\mathbf{P}\text{-a.s.}) \quad \Longrightarrow \quad 0 \in R(U) \tag{16}$$

(we invite to compare (16) with (9)).

Implication (16) exactly coincides with what is termed as Axiom $\mathbf{A1}_s$ in [27, Section 6], which addresses the coherent aggregation of random portfolios. This condition plays a pivotal role in preserving the coherence of the aggregation process, hence ensuring that the resulting composite map adheres to the defining properties of a risk measure, even after aggregating the underlying random portfolios. Reference is made to point (ii) in [27, Theorem 6.1] for further specifics.

Proceeding with the review, a set-valued risk measure $R(\cdot)$ associated with K is said to be: *convex* if, and only if, for any $U, V \in L^\infty_d(\Omega)$ and $\lambda \in]0,1[$,

$$\lambda R(U) + (1-\lambda)R(V) \subseteq R\big(\lambda U + (1-\lambda)V\big); \tag{17}$$

subadditive if, and only if, for any $U, V \in L^\infty_d(\Omega)$,

$$R(U) + R(V) \subseteq R(U+V); \tag{18}$$

positively homogeneous if, and only if, for any $U \in L^\infty_d(\Omega)$ and $t \in]0,\infty[$,

$$R(tU) = tR(U); \tag{19}$$

coherent if, and only if, it is *subadditive and positively homogeneous.*

Convexity embodies the diversification principle, asserting that combining two portfolios results in a risk no greater than the weighted average of their individual risks. Subadditivity complements this by guaranteeing that the total risk of combined positions does not exceed the sum of their separate risks, thereby fostering risk-sharing or pooling strategies. Positive homogeneity addresses scalability, stating that increasing a portfolio by a positive factor proportionally amplifies its risk, maintaining consistency under linear transformations. Together, these properties form the foundation of coherence, ensuring that the risk measure aligns with intuitive principles of rationality and provides a robust framework for assessing and managing financial uncertainty.

If a set-valued risk measure $R(\cdot)$ associated with K is convex, then, for any $U \in L^\infty_d(\Omega)$, $R(U)$ is a convex subset of $\mathbb{R}^d$; if $R(\cdot)$ is positively homogeneous, then $R(0)$ is a (closed) cone in $\mathbb{R}^d$.

It is also noteworthy that, if $R(\cdot)$ is subadditive, then the equivalence

$$(R\textbf{-mon}_\mathbf{s}) \quad \Longleftrightarrow \quad (16)$$

holds: that is, (16) implies (11). To justify this, let $U, V \in L^\infty_d(\Omega)$ satisfy $V - U \in R(0)$ ($\mathbf{P}$-a.s.). Thus, we aim to establish that $R(U) \subseteq R(V)$. By virtue of (16), we deduce that $0 \in R(V-U)$, and consequently, from subadditivity (see (18)), we conclude that

$$R(U) \subseteq R(U) + R(V-U) \subseteq R(V).$$

Finally, we present what might, at first glance, be regarded as a natural extension of the concept of a set-valued risk measure $R(\cdot)$ associated with K. Nonetheless, it merits emphasis that this new mapping does not constitute a straightforward generalization of $R(\cdot)$.

Indeed, it intrinsically necessitates a specific choice of the m-dimensional subspace M, that is,

$$M = \mathbb{R}^m \equiv \Big\{ \tilde{u} = [\tilde{u}_1, \ldots, \tilde{u}_n]^\mathsf{T} \in \mathbb{R}^n \ \Big|\ \forall\, i = m+1, \ldots, n,\ \tilde{u}_i = 0 \Big\}, \tag{20}$$

as given in (1). This identification aligns with the background typically adopted in the set-valued risk measure literature, as notably articulated in [14], especially at the beginning of Section 2 thereof. In that work, such a linear structure is explicitly imposed and carefully discussed.

This specification is not merely conventional but rather essential for the formulation of condition **($\widetilde{R}$-inv)** below, which governs the translation behavior under this setting (see in particular (21)). Accordingly, to define this notion, we proceed by replacing d with n, and K with $\widetilde{K}$, within the original framework built around $R(\cdot)$ (see also Remark 1.1 for further clarification).

In precise terms, we then introduce a *set-valued risk measure associated with (M and)* $\widetilde{K}$ as a function

$$\widetilde{R}\colon L_n^\infty(\Omega) \to \mathbb{F}_M$$

which satisfies the following three properties (the definition of $\mathbb{F}_M$ is provided in (3)).

($\widetilde{R}$-nor) — *normalization* (*w.r.t.* $\widetilde{K}$ *and* M):

$$\widetilde{K} \cap M \equiv K \cap M \subseteq \widetilde{R}(0), \qquad \widetilde{R}(0) \cap -\operatorname{int}\bigl(\widetilde{K} \cap M\bigr) = \emptyset$$

(refer to (2)).

($\widetilde{R}$-inv) — *translation invariance in* M: for any $X \in L_n^\infty(\Omega)$ and $u \in M$,

$$\widetilde{R}(X + \tilde{u}) = \widetilde{R}(X) - u$$

where we utilize the following notation: for any $v \in \mathbb{R}^d$,

$$\tilde{v} := [v, 0, \ldots, 0]^\mathsf{T} \in \mathbb{R}^n. \tag{21}$$

($\widetilde{R}$-mon) — *monotonicity* (*w.r.t.* $\widetilde{K}$): for any $X, Y \in L_n^\infty(\Omega)$,

$$X \leq_{\widetilde{K}} Y \ (\mathbf{P}\text{-a.s.}) \quad \Longrightarrow \quad \widetilde{R}(X) \subseteq \widetilde{R}(Y).$$

Similarly to the previous definition, the *effective domain* of $\widetilde{R}(\cdot)$ is the set

$$\operatorname{dom}\bigl(\widetilde{R}\bigr) \doteq \Bigl\{X \in L_n^\infty(\Omega) \Bigm| \widetilde{R}(X) \neq \emptyset\Bigr\}.$$

Clearly, if $n = d$ and $\widetilde{K} = K$, then the two definitions of set-valued risk measure coincide.

We also introduce two additional normalization conditions regarding $\widetilde{R}(\cdot)$, in line with those discussed before for $R(\cdot)$ (keeping (2) in mind).

($\widetilde{R}$-nor$_\mathbf{s}$) — *strong normalization* (*w.r.t.* $\widetilde{K}$ *and* M):

$$\widetilde{K} \cap M \equiv K \cap M = \widetilde{R}(0). \tag{22}$$

($\widetilde{R}$-nor*) — $\widetilde{R}(0)$-*additive invariance*: $\widetilde{R}(0) \notin \{\emptyset, M\}$ and, for any $X \in L_n^\infty(\Omega)$,

$$\widetilde{R}(X) + \widetilde{R}(0) = \widetilde{R}(X).$$

For a set-valued risk measure $\widetilde{R}\colon L_n^\infty(\Omega) \to \mathbb{F}_M$ associated with $\widetilde{K}$ the notions of *convexity*, *subadditivity*, *positive homogeneity* and *coherence* extend in a canonical way.

Remark 1.5. For $d = 1$, a set-valued risk measure $\widetilde{R}(\cdot)$ associated with $\widetilde{K}$ can be interpreted as a generalization of what is termed an *insensitive systemic risk measure*, as described in [1].

It remains to note that all subsequent statements independent of **($\widetilde{R}$-inv)** entail, by construction, no restrictions on the configuration of the subspace M—such as, for instance, that in (20).

1.2 Risk measures for portfolio aggregation

We can now propose our notion of *portfolio aggregation*, substantially adapting it from [27, Definition 6.1] to fit our more general setting (see [28] for additional insights). As in this work, the axioms we establish guarantee that, under relatively mild assumptions, a risk measure composed with an aggregator continues to satisfy the properties of a valid risk measure (refer primarily to Proposition 1.1). Nevertheless, we assign these postulates a specific denomination, one that aims to be inherently suggestive of the underlying intuition guiding their articulation. All such conditions will naturally be framed in relation to the reference set $R(0)$.

Definition 1.1. Let $R\colon L^\infty_d(\Omega) \to \mathbb{F}_M$ be a set-valued risk measure associated with K. An *R-portfolio aggregator associated with ($R(\cdot)$ and) $\widetilde{K}$* is a measurable (deterministic) *bounded* function

$$f\colon \mathbb{R}^n \to \mathbb{R}^d$$

such that the following four axioms hold.

(f-com) — *$R(0)$-comparability (w.r.t. $\widetilde{K}$)*:

$$f\big(\widetilde{K}\big) \subseteq R(0).$$

(f-nor) — *normalization (w.r.t. $\widetilde{K}$)*:

$$f(0) \in R(0) \cap -R(0).$$

(f-inv) — *translation invariance in M*: for any $x \in \mathbb{R}^n$ and $u \in M$,

$$f(x+\tilde{u}) - f(x) - u \in R(0) \cap -R(0),$$

where $\tilde{u}$ is specified as expressed by (21), that is, $\tilde{u} = [u, 0, \dots, 0]^\mathsf{T} \in \mathbb{R}^n$.

(f-mon) — *monotonicity (w.r.t. $\widetilde{K}$)*: for any $x, y \in \mathbb{R}^n$,

$$x \leq_{\widetilde{K}} y \quad \Longrightarrow \quad f(y) - f(x) \in R(0).$$

In comparing our Definition 1.1 with the aforementioned [27, Definition 6.1], we highlight that the first and fourth axioms, **(f-com)** and **(f-inv)**, directly correspond to the analogous statements **PA1** and **PA4** in [27, Definition 6.1]. Additionally, Axiom **PA2** in [27, Definition 6.1] will be captured by our **(f-sub)**, as introduced later (*subadditivity*), while Axiom **PA3** in [27, Definition 6.1] aligns with our **(f-hom)** further below (*positive homogeneity*). It is important to stress that **(f-nor)** and **(f-mon)** hold crucial relevance within the broader framework we are developing, as previously anticipated in our discussion.

Before advancing to the opening results of our paper, let us briefly comment on Definition 1.1, also in connection with Section 1, starting with the consideration that, by **(f-nor)**,

$$\pm f(0) \in R(0). \tag{23}$$

Moreover, if (15) holds for $R(\cdot)$ (which is the case, for instance, when **(R-nor$_\mathbf{s}$)** is satisfied), then the equivalence

$$\textbf{(f-nor)} \quad \Longleftrightarrow \quad f(0) = 0$$

is clearly verified and this renders **(f-com)** and (23) superfluous (see also (8)).

Our first fundamental finding is stated and proven immediately thereafter.

Proposition 1.1. *Let* $R\colon L_d^\infty(\Omega) \to \mathbb{F}_M$ *be a set-valued risk measure associated with* K *and let* $f\colon \mathbb{R}^n \to \mathbb{R}^d$ *be an* R*-portfolio aggregator associated with* $\widetilde{K}$*. Define the map* $\widetilde{R}\colon L_n^\infty(\Omega) \to \mathbb{F}_M$ *by*

$$\widetilde{R}(X) := R(f)(X) \doteq R\big(f(X)\big), \quad X \in L_n^\infty(\Omega). \tag{24}$$

Assume that **(**R**-mon$_\mathbf{s}$)** *holds for* $R(\boldsymbol{\cdot})$*. Then,* $\widetilde{R}(\boldsymbol{\cdot})$ *is a set-valued risk measure associated with* $\widetilde{K}$*, fulfilling the following identity:*

$$\widetilde{R}(0) = R(0). \tag{25}$$

We refer to $\widetilde{R}(\boldsymbol{\cdot})$ *as a* set-valued risk measure for portfolio aggregation associated with K and $\widetilde{K}$.

Proof. The demonstration relies on illustrating that the function $\widetilde{R}(\boldsymbol{\cdot})$, defined in (24), satisfies axioms **(**$\widetilde{R}$**-nor)**, **(**$\widetilde{R}$**-inv)** and **(**$\widetilde{R}$**-mon)**, which characterize a set-valued risk measure associated with $\widetilde{K}$. This is accomplished by building upon the analogous axioms of $R(\boldsymbol{\cdot})$ as a set-valued risk measure associated with K, where **(**R**-mon$_\mathbf{s}$)** is presumed in place of **(**R**-mon)**, and the properties of $f(\boldsymbol{\cdot})$ as an R-portfolio aggregator associated with $\widetilde{K}$ (see Definition 1.1).

To begin, we verify that $\widetilde{R}(\boldsymbol{\cdot})$ satisfies (25), and therefore, by (2) and **(**R**-nor)**, **(**$\widetilde{R}$**-nor)**. By (23),

$$\begin{cases} f(0) - 0 \equiv f(0) \in R(0), \\ 0 - f(0) \equiv -f(0) \in R(0). \end{cases}$$

Then, by **(**R**-mon$_\mathbf{s}$)**,

$$\widetilde{R}(0) \equiv R\big(f(0)\big) = R(0).$$

Next, we prove that $\widetilde{R}(\boldsymbol{\cdot})$ satisfies **(**$\widetilde{R}$**-inv)**. Let $X \in L_n^\infty(\Omega)$ and $u \in M$. Our objective is to derive that

$$\widetilde{R}(X + \tilde{u}) = \widetilde{R}(X) - u.$$

By **(**f**-inv)**,

$$f(X + \tilde{u}) - f(X) - u \in R(0) \cap -R(0)$$

(**P**-a.s.). In particular, $f(X + \tilde{u}) - \big[f(X) + u\big] \in R(0)$ (**P**-a.s.). Then, by **(**R**-inv)** and **(**R**-mon$_\mathbf{s}$)**,

$$\widetilde{R}(X) - u \equiv R\big(f(X)\big) - u = R\big(f(X) + u\big) \subseteq R\big(f(X + \tilde{u})\big) \equiv \widetilde{R}(X + \tilde{u}).$$

Analogously, $f(X) + u - f(X + \tilde{u}) \in R(0)$ (**P**-a.s.), and then, by **(**R**-mon$_\mathbf{s}$)** and **(**R**-inv)** again,

$$\widetilde{R}(X + \tilde{u}) \equiv R\big(f(X + \tilde{u})\big) \subseteq R\big(f(X) + u\big) = R\big(f(X)\big) - u \equiv \widetilde{R}(X) - u.$$

Finally, we establish that $\widetilde{R}(\boldsymbol{\cdot})$ satisfies **(**$\widetilde{R}$**-mon)**. Let $X, Y \in L_n^\infty(\Omega)$ such that $X \leq_{\widetilde{K}} Y$ (**P**-a.s.). Let us show that

$$\widetilde{R}(X) \subseteq \widetilde{R}(Y).$$

By **(**f**-mon)**,

$$f(Y) - f(X) \in R(0)$$

(**P**-a.s.). Then, by **(**R**-mon$_\mathbf{s}$)**,

$$\widetilde{R}(X) \equiv R\big(f(X)\big) \subseteq R\big(f(Y)\big) \equiv \widetilde{R}(Y). \qquad \square$$

Remark 1.6. Referring to Proposition 1.1 and, specifically, to (25), it can be readily seen that, if **(**R**-nor$_\mathbf{s}$)** holds for $R(\boldsymbol{\cdot})$, then **(**$\widetilde{R}$**-nor$_\mathbf{s}$)** holds for $\widetilde{R}(\boldsymbol{\cdot})$, by relating (10) and (22). More generally, still based on (25), **(**R**-nor*)** would imply **(**$\widetilde{R}$**-nor*)** (see (24)). Additionally, as a straightforward consequence of (25) and (8), we have

$$0 \in \widetilde{R}(0).$$

The function $\widetilde{R}(\cdot) \equiv R(f)(\cdot)$, defined as in (24), can be shown to possess properties as convexity, subadditivity, positive homogeneity, or even coherence as a set-valued risk measure associated with $\widetilde{K}$, provided that the set-valued risk measure $R(\cdot)$, associated with K, and the R-portfolio aggregator $f(\cdot)$, associated with $\widetilde{K}$, exhibit enhanced regularity beyond the requirements of their respective foundational definitions. In view of this, we introduce the following definition.

Definition 1.2. Let $R\colon L_d^\infty(\Omega) \to \mathbb{F}_M$ be a set-valued risk measure associated with K. An R-portfolio aggregator $f(\cdot)$ associated with $\widetilde{K}$ (see Definition 1.1) is a R-*concave portfolio aggregator* if the following condition is satisfied.

(f-con) — *concavity*: for any $x, y \in \mathbb{R}^n$ and $\lambda \in]0,1[$,

$$f\big(\lambda x + (1-\lambda)y\big) - \lambda f(x) - (1-\lambda) f(y) \in R(0).$$

Furthermore, $f(\cdot)$ is a R-*coherent portfolio aggregator* if the following two properties are fulfilled.

(f-sub) — *subadditivity*: for any $x, y \in \mathbb{R}^n$,

$$f(x+y) - f(x) - f(y) \in R(0).$$

(f-hom) — *positive homogeneity*: for any $x \in \mathbb{R}^n$ and $t \in]0,\infty[$,

$$f(tx) - t f(x) \in R(0) \cap -R(0).$$

Remark 1.7. Regarding Definition 1.2, we observe that, if $R(\cdot)$ is subadditive (see (18)) and $f(\cdot)$ is a R-coherent portfolio aggregator, then $f(\cdot)$ is a R-concave portfolio aggregator (see **(f-con)**). Indeed, by **(f-sub)** and **(f-hom)**, it turns out that, for any $x, y \in \mathbb{R}^n$ and $\lambda \in]0,1[$, the vector

$$\begin{aligned} f\big(\lambda x + (1-\lambda)y\big) &- \lambda f(x) - (1-\lambda) f(y) \\ &\equiv f\big(\lambda x + (1-\lambda)y\big) - f(\lambda x) - f\big((1-\lambda)y\big) \\ &\qquad + f(\lambda x) - \lambda f(x) + f\big((1-\lambda)y\big) - (1-\lambda) f(y) \end{aligned}$$

belongs to $R(0)$, as also supported by (14).

Here is the expected regularity outcome we alluded to earlier.

Proposition 1.2. *Let $R\colon L_d^\infty(\Omega) \to \mathbb{F}_M$ be a set-valued risk measure associated with K, which attains* **(R-mon$_\mathbf{s}$)**, *and let $f\colon \mathbb{R}^n \to \mathbb{R}^d$ be an R-portfolio aggregator associated with $\widetilde{K}$ such that the function*

$$\widetilde{R} \coloneqq R(f)\colon L_n^\infty(\Omega) \to \mathbb{F}_M,$$

defined as in (24), *is a set-valued risk measure (associated with $\widetilde{K}$). Then, the following applies.*

1. *If $R(\cdot)$ is convex and $f(\cdot)$ is R-concave, then $\widetilde{R}(\cdot)$ is convex.*

2. *If $R(\cdot)$ is subadditive and* **(f-sub)** *holds (for $f(\cdot)$), then $\widetilde{R}(\cdot)$ is subadditive.*

3. *If $R(\cdot)$ is positively homogeneous and* **(f-hom)** *holds, then $\widetilde{R}(\cdot)$ is positively homogeneous.*

4. *If $R(\cdot)$ is coherent and $f(\cdot)$ is R-coherent, then $\widetilde{R}(\cdot)$ is coherent.*

Proof. We show, step by step, that $\widetilde{R}(\cdot)$ is convex, subadditive, positively homogeneous, and coherent, contingent upon the proper structure of $R(\cdot)$ and $f(\cdot)$. The key definitions to keep in mind for this are Definitions 1.1 and 1.2.

1. Let $X, Y \in L_n^\infty(\Omega)$ and $\lambda \in \,]0,1[$. We seek to establish that

$$\lambda \widetilde{R}(X) + (1-\lambda)\widetilde{R}(Y) \subseteq \widetilde{R}\big(\lambda X + (1-\lambda)Y\big).$$

By **(f-con)**,

$$f\big(\lambda X + (1-\lambda)Y\big) - \big[\lambda f(X) + (1-\lambda) f(Y)\big] \in R(0)$$

(**P**-a.s.). Then, by convexity of $R(\boldsymbol{\cdot})$ (see (17)) and **(R-mon$_\mathbf{s}$)**,

$$\begin{aligned}\lambda \widetilde{R}(X) + (1-\lambda)\widetilde{R}(Y) &\equiv \lambda R\big(f(X)\big) + (1-\lambda) R\big(f(Y)\big)\\ &\subseteq R\big(\lambda f(X) + (1-\lambda) f(Y)\big)\\ &\subseteq R\big(f\big(\lambda X + (1-\lambda)Y\big)\big)\\ &\equiv \widetilde{R}\big(\lambda X + (1-\lambda)Y\big).\end{aligned}$$

2. Let $X, Y \in L_n^\infty(\Omega)$. Let us verify that

$$\widetilde{R}(X) + \widetilde{R}(Y) \subseteq \widetilde{R}(X+Y).$$

By **(f-sub)**,

$$f(X+Y) - \big[f(X) + f(Y)\big] \in R(0)$$

(**P**-a.s.). Then, by subadditivity of $R(\boldsymbol{\cdot})$ (see (18)) and **(R-mon$_\mathbf{s}$)**,

$$\widetilde{R}(X) + \widetilde{R}(Y) \equiv R\big(f(X)\big) + R\big(f(Y)\big) \subseteq R\big(f(X) + f(Y)\big) \subseteq R\big(f(X+Y)\big) \equiv \widetilde{R}(X+Y).$$

3. Let $X, Y \in L_n^\infty(\Omega)$ and $t \in \,]0,\infty[$. Our purpose is to demonstrate that

$$\widetilde{R}(tX) = t\widetilde{R}(X).$$

By **(f-hom)**,

$$\pm\big[f(tX) - tf(X)\big] \in R(0)$$

(**P**-a.s.). Therefore, by applying **(R-mon$_\mathbf{s}$)** twice,

$$R\big(f(tX)\big) = R\big(tf(X)\big).$$

Then, also by positive homogeneity of $R(\boldsymbol{\cdot})$ (see (19)),

$$\widetilde{R}(tX) \equiv R\big(f(tX)\big) = R\big(tf(X)\big) = tR\big(f(X)\big) \equiv t\widetilde{R}(X).$$

4. This assertion follows as an immediate corollary of points **2** and **3**, just proved. □

We bring the current (sub)section to a close with our key findings on set-valued risk measures for portfolio aggregation regarding *maximal correlation* (see Propositions 1.1 and 1.2). Prior to this, for the sake of self-consistency (within the paper), we include a lemma that addresses two basic equalities between sets. This lemma is fairly simple and does not require a formal proof.

Lemma 1.1. *Let A be a set and, for all $\alpha \in A$, let $U_\alpha \subseteq \mathbb{R}^d$. The following two identities hold.*

1. *For any $u \in \mathbb{R}^d$,*

$$\bigcap_{\alpha\in A} \Big\{U_\alpha - u\Big\} = \left\{\bigcap_{\alpha\in A} U_\alpha\right\} - u.$$

2. *For any $t \in \,]0,\infty[$,*

$$\bigcap_{\alpha\in A} \Big\{tU_\alpha\Big\} = t\left\{\bigcap_{\alpha\in A} U_\alpha\right\}.$$

We now present the concluding result of this section.

Proposition 1.3. *Let A be a nonempty set and, for all $\alpha \in A$, $\widetilde{R}_\alpha \colon L_n^\infty(\Omega) \to \mathbb{F}_M$ be a set-valued risk measure associated with $\widetilde{K}$. Define the map $R_A \colon L_n^\infty(\Omega) \to \mathbb{F}_M$ by*

$$R_A(X) \doteq \bigcap_{\alpha \in A} \widetilde{R}_\alpha(X), \quad X \in L_n^\infty(\Omega). \tag{26}$$

Then $R_A(\cdot)$ is a set-valued risk measure associated with $\widetilde{K}$, satisfying the following conditions.

1. *If there exists $\bar{\alpha} \in A$ such that $\widetilde{R}_{\bar{\alpha}}(\cdot)$ satisfies ($\widetilde{R}$-**nor**$_\mathbf{s}$), then $R_A(\cdot)$ satisfies ($\widetilde{R}$-**nor**$_\mathbf{s}$).*

2. *If, for all $\alpha \in A$, $\widetilde{R}_\alpha(\cdot)$ satisfies ($\widetilde{R}$-**nor***), then $R_A(\cdot)$ satisfies ($\widetilde{R}$-**nor***).*

3. *If, for all $\alpha \in A$, $\widetilde{R}_\alpha(\cdot)$ is convex, then $R_A(\cdot)$ is convex.*

4. *If, for all $\alpha \in A$, $\widetilde{R}_\alpha(\cdot)$ is subadditive, then $R_A(\cdot)$ is subadditive.*

5. *If, for all $\alpha \in A$, $\widetilde{R}_\alpha(\cdot)$ is positively homogeneous, then $R_A(\cdot)$ is positively homogeneous.*

6. *If, for all $\alpha \in A$, $\widetilde{R}_\alpha(\cdot)$ is coherent, then $R_A(\cdot)$ is coherent.*

We call $R_A(\cdot)$ a maximal correlation set-valued risk measure.

Proof. Initially, we verify that $R_A(\cdot)$, as defined in (26), fulfils axioms ($\widetilde{R}$-**nor**), ($\widetilde{R}$-**inv**), and ($\widetilde{R}$-**mon**). This emerges from the fact that, for all $\alpha \in A$, $\widetilde{R}_\alpha(\cdot)$ is assumed to be a set-valued risk measure (associated with $\widetilde{K}$). Subsequently, we establish that $R_A(\cdot)$ also inherits the potential properties of convexity, subadditivity, positive homogeneity, and coherence from the maps $\widetilde{R}_\alpha(\cdot)$.

To begin, we show that $R_A(\cdot)$ satisfies ($\widetilde{R}$-**nor**), namely, that

$$\widetilde{K} \cap M \subseteq R_A(0), \quad R_A(0) \cap -\operatorname{int}\big(\widetilde{K} \cap M\big) = \emptyset.$$

Since each $\widetilde{R}_\alpha(\cdot)$ satisfies ($\widetilde{R}$-**nor**), it follows that, for all $\alpha \in A$,

$$\widetilde{K} \cap M \subseteq \widetilde{R}_\alpha(0), \quad \widetilde{R}_\alpha(0) \cap -\operatorname{int}\big(\widetilde{K} \cap M\big) = \emptyset.$$

In light of (26), this immediately leads us to the thesis.

Next, we validate that $R_A(\cdot)$ satisfies ($\widetilde{R}$-**inv**). Let $X \in L_n^\infty(\Omega)$ and $u \in M$. What we intend to achieve is that

$$R_A(X + \tilde{u}) = R_A(X) - u.$$

As each $\widetilde{R}_\alpha(\cdot)$ satisfies ($\widetilde{R}$-**inv**), it results that, for all $\alpha \in A$,

$$\widetilde{R}_\alpha(X + \tilde{u}) = \widetilde{R}_\alpha(X) - u.$$

Then, also by point **1** of Lemma 1.1,

$$R_A(X + \tilde{u}) \equiv \bigcap_{\alpha \in A} \widetilde{R}_\alpha(X + \tilde{u}) = \bigcap_{\alpha \in A} \Big\{ \widetilde{R}_\alpha(X) - u \Big\} = \left\{ \bigcap_{\alpha \in A} \widetilde{R}_\alpha(X) \right\} - u \equiv R_A(X) - u.$$

Finally, we demonstrate that $R_A(\cdot)$ satisfies ($\widetilde{R}$-**mon**). Let $X, Y \in L_n^\infty(\Omega)$ such that $X \leq_{\widetilde{K}} Y$ (**P**-a.s.). We want to obtain that

$$R_A(X) \subseteq R_A(Y).$$

Since each $\widetilde{R}_\alpha(\cdot)$ satisfies ($\widetilde{R}$-**mon**), it follows that, for all $\alpha \in A$,

$$\widetilde{R}_\alpha(X) \subseteq \widetilde{R}_\alpha(Y).$$

Then, from (26), for all $\alpha \in A$,

$$R_A(X) \subseteq \widetilde{R}_\alpha(Y).$$

The claim arises from the arbitrariness of the parameter $\alpha \in A$.

We now examine the regularity properties of $R_A(\cdot)$ that may be induced by $R_\alpha(\cdot)$, $\alpha \in A$.

1. Let $\bar{\alpha} \in A$ be such that
$$\widetilde{R}_{\bar{\alpha}}(0) = \widetilde{K} \cap M.$$
Then, directly from (26), we infer that
$$R_A(0) \subseteq \widetilde{K} \cap M.$$
This is sufficient to attain **($\widetilde{R}$-nor$_\mathbf{s}$)** for $R_A(\cdot)$, since the reverse inclusion is guaranteed by **($\widetilde{R}$-nor)**, which has already been substantiated.

2. Let $X \in L_n^\infty(\Omega)$. We wish to ascertain that
$$R_A(X) + R_A(0) = R_A(X).$$
As each $\widetilde{R}_\alpha(\cdot)$ satisfies **($\widetilde{R}$-nor)**, it results that, for all $\alpha \in A$, $0 \in \widetilde{R}_\alpha(0)$, and then, from (26),
$$0 \in R_A(0).$$
Therefore, the desired statement is equivalent solely to the set inclusion
$$R_A(X) + R_A(0) \subseteq R_A(X)$$
(as already argued throughout Subsection 1.1). This inclusion holds true because each $\widetilde{R}_\alpha(\cdot)$ satisfies **($\widetilde{R}$-nor*)**, which allows us to write
$$R_A(X) + R_A(0) \equiv \bigcap_{\alpha \in A} \widetilde{R}_\alpha(X) + \bigcap_{\alpha \in A} \widetilde{R}_\alpha(0) \subseteq \bigcap_{\alpha \in A} \Big[\widetilde{R}_\alpha(X) + \widetilde{R}_\alpha(0)\Big] = \bigcap_{\alpha \in A} \widetilde{R}_\alpha(X) \equiv R_A(X).$$

3. Let $X, Y \in L_n^\infty(\Omega)$ and $\lambda \in]0,1[$. We want to determine that
$$\lambda R_A(X) + (1-\lambda) R_A(Y) \subseteq R_A\big(\lambda X + (1-\lambda)Y\big).$$
Since each $\widetilde{R}_\alpha(\cdot)$ is convex, it follows that, for all $\alpha \in A$,
$$\lambda \widetilde{R}_\alpha(X) + (1-\lambda) \widetilde{R}_\alpha(Y) \subseteq \widetilde{R}_\alpha\big(\lambda X + (1-\lambda)Y\big).$$
Then, from (26) (reasoning separately on each summand), for all $\alpha \in A$,
$$\lambda R_A(X) + (1-\lambda) R_A(Y) \subseteq \widetilde{R}_\alpha\big(\lambda X + (1-\lambda)Y\big).$$
The result is derived by intersecting over the set A on the right-hand side.

4. Let $X, Y \in L_n^\infty(\Omega)$. Let us prove that
$$R_A(X) + R_A(Y) \subseteq R_A(X+Y).$$
As each $\widetilde{R}_\alpha(\cdot)$ is subadditive, it results that, for all $\alpha \in A$,
$$\widetilde{R}_\alpha(X) + \widetilde{R}_\alpha(Y) \subseteq \widetilde{R}_\alpha(X+Y).$$
Then, from (26), for all $\alpha \in A$,
$$R_A(X) + R_A(Y) \subseteq \widetilde{R}_\alpha(X+Y),$$
and we reach the conclusion in the same way as before.

5. Let $X \in L_n^\infty(\Omega)$ and $t \in \,]0,\infty[$. We aim to deduce that

$$R_A(tX) = tR_A(X).$$

Since each $\widetilde{R}_\alpha(\cdot)$ is positively homogeneous, it follows that, for all $\alpha \in A$,

$$\widetilde{R}_\alpha(tX) = t\widetilde{R}_\alpha(X).$$

Then, also by point **2** of Lemma 1.1,

$$R_A(tX) \equiv \bigcap_{\alpha\in A} \widetilde{R}_\alpha(tX) = \bigcap_{\alpha\in A} \left\{t\widetilde{R}_\alpha(X)\right\} = t\left\{\bigcap_{\alpha\in A} \widetilde{R}_\alpha(X)\right\} \equiv tR_A(X).$$

6. This is a straightforward outcome of points **4** and **5**. □

What is established in Proposition 1.3 applies, in particular, when each set-valued risk measure $\widetilde{R}_\alpha(\cdot)$—which, according to (26), constitutes the maximal correlation set-valued risk measure for portfolio aggregation $R_A(\cdot)$—is taken as a composite risk measure. More precisely, if, for all $\alpha \in A$, there exists a set-valued risk measure $R_\alpha\colon L_d^\infty(\Omega) \to \mathbb{F}_M$ associated with K, along with an R_α-portfolio aggregator $f_\alpha\colon \mathbb{R}^n \to \mathbb{R}^d$ associated with $\widetilde{K}$ (see Definition 1.1), such that

$$\widetilde{R}_\alpha = R_\alpha(f_\alpha) \tag{27}$$

as defined in (24), then the maximal correlation risk measure $R_A(\cdot)$ assumes the form

$$R_A(X) = \bigcap_{\alpha\in A} R_\alpha\big(f_\alpha(X)\big), \quad X \in L_n^\infty(\Omega) \tag{28}$$

(refer to (26)), and Proposition 1.3 remains valid in this setting. In this case, $R_A(\cdot)$ may specifically be termed a *maximal correlation set-valued risk measure for portfolio aggregation.*

Furthermore, if, for all $\alpha \in A$, $R_\alpha(\cdot)$ satisfies the stronger monotonicity condition **(R-mon$_\mathbf{s}$)** rather than merely **(R-mon)**, then Proposition 1.1 guarantees that $\widetilde{R}_\alpha(\cdot)$ qualifies as a set-valued risk measure (associated with $\widetilde{K}$), while also ensuring that

$$\widetilde{R}_\alpha(0) = R_\alpha(0)$$

(see (25)). Additionally, Remark 1.6 and Proposition 1.2 provide a comprehensive characterization of how the regularity properties of any composite risk measure $\widetilde{R}_\alpha(\cdot)$ depend on the structural properties of $R_\alpha(\cdot)$ and $f_\alpha(\cdot)$ (see also Definition 1.2).

2 Consistency under stochastic orderings of risks

In this section, we examine the *consistency* of set-valued functions with respect to various classes of multivariate *stochastic orderings*, structured similarly to those in Section 1 (see (26), Proposition 1.3, and (28)). Specifically, we consider maps $\Psi_A\colon L_n^\infty(\Omega) \to \mathcal{P}\big(\mathbb{R}^d\big)$ defined as

$$\Psi_A(X) \doteq \bigcap_{\alpha\in A} \Psi_\alpha\big(g_\alpha(X)\big), \quad X \in L_n^\infty(\Omega) \tag{29}$$

where A is a nonempty set and, for all $\alpha \in A$, $\Psi_\alpha\colon L_d^\infty(\Omega) \to \mathcal{P}\big(\mathbb{R}^d\big)$ is a set-valued function itself (with $\mathcal{P}\big(\mathbb{R}^d\big)$ denoting the power set of $\mathbb{R}^d$) and $g_\alpha\colon \mathbb{R}^n \to \mathbb{R}^d$ is a measurable and bounded function.

In this regard, we would like to highlight a core aspect that establishes a direct connection with the concepts provided in the preceding section. For all $\alpha \in A$, let $R_\alpha\colon L_d^\infty(\Omega) \to \mathbb{F}_M$ be a set-valued risk measure associated with K, $f_\alpha \colon \mathbb{R}^n \to \mathbb{R}^d$ be an R_α-portfolio aggregator associated with $\widetilde{K}$, and let $R_A \colon L^\infty_n(\Omega) \to \mathbb{F}_M$ be settled as in (28). If we set, for all $\alpha \in A$,

$$\Psi_\alpha(U) \coloneqq R_\alpha(-U), \qquad g_\alpha(x) \coloneqq -f_\alpha(-x)$$

(with $U \in L^\infty_d(\Omega)$ and $x \in \mathbb{R}^n$), then, from (28) and (29), it follows that, for any $X \in L^\infty_n(\Omega)$,

$$\Psi_A(X) = R_A(-X).$$

Please refer again to [5] to identify analogous approaches or comparable mechanisms.

In this analysis, we aim to maintain flexibility by not strictly adhering to all the axioms typically associated with set-valued risk measures and portfolio aggregators (see Section 1), as these are not essential to the technical contributions of this section. We may therefore regard such functionals as "*pseudo*" or "*improper*" risk measures, in the sense that they do not satisfy all the axioms outlined in Subsection 1.1—neither the fundamental ones nor those imposing stronger regularity conditions (such as convexity, subadditivity, positive homogeneity, or coherence).

Nonetheless, the literature itself embraces a broader notion: mappings that fulfil only part of the usual axioms are still termed risk measures. See, for instance, [25], and the recent developments beyond cash-(sub)additivity in [30, 31]. Related contributions that retain a more classical structure include, for example, [8].

2.1 Stochastic orderings via classes of functions

To address the consistency of set-valued risk measures with respect to a stochastic ordering, we build upon the setting presented in [5], employing the *set-valued upper expectation* operator $\mathbf{E}^+[\,\cdot\,]$.

The introduction of this operator in [23], and later in [22], has laid a robust foundation for analyzing the desired properties of a set-valued risk measure, offering a refined approach to their investigation. Precisely, [23] focuses on the *dual representation* of set-valued coherent risk measures within *conical* market models, while [22] delves into a general framework for set-valued *T-translative* functions. This operator has been demonstrated to deliver a more comprehensive evaluation of risk associated with financial positions, considering the entire set of possible outcomes.

Stochastic orderings are also defined concerning a broad class of functions, $\mathcal{F}^{d,d}$ or $\mathcal{F}^{n,d}$, depending on the specific context. In particular, Subsection 2.2 is dedicated to a detailed study of the class $\mathcal{F}^{n,d}_{\prec}$ of *Schur-convex* functions (we refer to the subsequent text for the precise definition of this class). For a thorough overview of stochastic orderings and their role in multivariate risk assessment, readers are encouraged to consult [4] and the references therein.

Accordingly, we first take into account the (*positive*) *dual* or *polar cone* of K, i.e.,

$$K^+ \coloneqq \left\{ w \in \mathbb{R}^d \;\middle|\; \forall\, z \in K,\, w^\mathsf{T} z \geq 0 \right\} \tag{30}$$

(a closed and convex cone containing the origin of $\mathbb{R}^d$), and thus, for any $w \in K^+ \setminus \{0\}$, the closed and homogeneous *half-space* with normal w, indicated by

$$H^+(w) \coloneqq \left\{ z \in \mathbb{R}^d \;\middle|\; 0 \leq w^\mathsf{T} z \right\}.$$

We remind in this respect that, for any $u, v \in \mathbb{R}^d$,

$$u \leq_K v \iff \forall\, w \in K^+,\, w^\mathsf{T} u \leq w^\mathsf{T} v \iff \forall\, w \in K^+,\, u \leq_{H^+(w)} v.$$

Now, we introduce

$$\mathcal{M}_1^{\mathbf{P}} \coloneqq \left\{ \mathbf{Q} \colon \mathcal{F} \to [0,1] \;\middle|\; \mathbf{Q} \text{ is a probability measure on } (\Omega, \mathcal{F}) \text{ with } \mathbf{Q} \ll \mathbf{P} \right\}$$

and, for any $w \in K^+ \setminus \{0\}$, we define

$$\mathcal{Q}(w) \equiv \mathcal{Q}_K^{\mathbf{P}}(w) \coloneqq \left\{ \mathbf{Q} = [\mathbf{Q}_1, \ldots, \mathbf{Q}_d]^\mathsf{T} \in \left(\mathcal{M}_1^{\mathbf{P}}\right)^d \;\middle|\; \left[w_1 \frac{d\mathbf{Q}_1}{d\mathbf{P}}, \ldots, w_d \frac{d\mathbf{Q}_d}{d\mathbf{P}} \right]^\mathsf{T} \in K^+ \ (\mathbf{P}\text{-a.s.}) \right\}. \tag{31}$$

For any $\mathbf{Q} = [\mathbf{Q}_1, \dots, \mathbf{Q}_d]^\mathsf{T} \in \left(\mathcal{M}_1^{\mathbf{P}}\right)^d$ and $Z = [Z_1, \dots, Z_d]^\mathsf{T} \in L_d^\infty(\Omega)$, we also denote

$$\mathbf{E}^{\mathbf{Q}}[Z] \doteq \Big[\mathbf{E}^{\mathbf{Q}_1}[Z_1], \dots, \mathbf{E}^{\mathbf{Q}_d}[Z_d]\Big]^\mathsf{T}.$$

In particular, for any measurable *bounded* (vector-valued) function $g = [g_1, \dots, g_d]^\mathsf{T}\colon \mathbb{R}^n \to \mathbb{R}^d$ and $X \in L_n^\infty(\Omega)$, it turns out that

$$\mathbf{E}^{\mathbf{Q}}\big[g(X)\big] = \Big[\mathbf{E}^{\mathbf{Q}_1}\big[g_1(X)\big], \dots, \mathbf{E}^{\mathbf{Q}_d}\big[g_d(X)\big]\Big]^\mathsf{T}$$

where, for any $i = 1, \dots, d$,

$$\mathbf{E}^{\mathbf{Q}_i}\big[g_i(X)\big] = \int_{\mathbb{R}^n} g_i(x)(\mathbf{Q}_i)^X(dx),$$

for $(\mathbf{Q}_i)^X$ being the distribution law of X (on $\mathbb{R}^n$) w.r.t. the probability measure $\mathbf{Q}_i$.

Finally, we consider the *set-valued upper expectation* operator on $L_d^\infty(\Omega)$ given by

$$\mathbf{E}^+[Z] \coloneqq \bigcap_{\substack{w \in K^+ \setminus \{0\} \\ \mathbf{Q} \equiv \mathbf{Q}_w \in \mathcal{Q}(w)}} \mathbf{E}^+_{\mathbf{Q},w}[Z], \quad Z \in L_d^\infty(\Omega)$$

where, for any $w \in K^+ \setminus \{0\}$ and $\mathbf{Q} \equiv \mathbf{Q}_w \in \mathcal{Q}(w)$,

$$\mathbf{E}^+_{\mathbf{Q},w}[Z] \coloneqq H^+(w) + \mathbf{E}^{\mathbf{Q}}[Z] = \left\{ z \in \mathbb{R}^d \;\middle|\; w^\mathsf{T}\mathbf{E}^{\mathbf{Q}}[Z] \le w^\mathsf{T} z \right\}. \tag{32}$$

Let $\mathcal{F}^{d,d}$ be an arbitrary (nonempty) class of measurable *bounded* functions from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $\mathcal{F}^{n,d}$ be an arbitrary class of measurable *bounded* functions from $\mathbb{R}^n$ to $\mathbb{R}^d$. The corresponding multivariate stochastic orderings, $\le_{\mathcal{F}^{d,d}}$ and $\le_{\mathcal{F}^{n,d}}$, are defined as follows, respectively.

Definition 2.1.

(i) For any $U, V \in L_d^\infty(\Omega)$,

$$U \le_{\mathcal{F}^{d,d}} V \quad \overset{\text{def}}{\iff} \quad \forall\, h \in \mathcal{F}^{d,d},\ \mathbf{E}^+\big[h(V)\big] \subseteq \mathbf{E}^+\big[h(U)\big].$$

(ii) For any $X, Y \in L_n^\infty(\Omega)$,

$$X \le_{\mathcal{F}^{n,d}} Y \quad \overset{\text{def}}{\iff} \quad \forall\, g \in \mathcal{F}^{n,d},\ \mathbf{E}^+\big[g(Y)\big] \subseteq \mathbf{E}^+\big[g(X)\big].$$

Definition 2.1 expands upon the concepts from [5], adapting them to our set-valued framework. To delineate specific classes of functions relevant for comparing the risk profiles of two random vectors, we introduce several additional notions.

Let $h\colon \mathbb{R}^d \to \mathbb{R}^d$ be any measurable function. Then, in relation to K, we classify $h(\cdot)$ as: *K-increasing* if, for any $u, v \in \mathbb{R}^d$,

$$u \le_K v \quad \Longrightarrow \quad h(u) \le_K h(v);$$

K-decreasing if, for any $u, v \in \mathbb{R}^d$,

$$u \le_K v \quad \Longrightarrow \quad h(v) \le_K h(u);$$

K-convex if, for any $u, v \in \mathbb{R}^d$ and $\lambda \in \,]0,1[$,

$$h\big(\lambda u + (1-\lambda)v\big) \le_K \lambda h(u) + (1-\lambda)h(v).$$

Among the possible classes of functions reflecting the above basic regularity behaviors, we focus on two of particular significance, specified by:

$$\mathcal{F}^{d,d}_{\mathrm{icx}} \equiv \mathcal{F}^{d,d}_{\mathrm{icx}}(K) \coloneqq \bigl\{ h\colon \mathbb{R}^d \to \mathbb{R}^d \bigm| h \text{ is bounded, } K\text{-increasing and } K\text{-convex} \bigr\}, \tag{33}$$

$$\mathcal{F}^{d,d}_{\mathrm{decx}} \equiv \mathcal{F}^{d,d}_{\mathrm{decx}}(K) \coloneqq \bigl\{ h\colon \mathbb{R}^d \to \mathbb{R}^d \bigm| h \text{ is bounded, } K\text{-decreasing and } K\text{-convex} \bigr\}. \tag{34}$$

To achieve consistency results for multifunctions of the form of $\Psi_A(\cdot)$ (see (29)), we adopt the following *closure* assumption for the chosen class $\mathcal{F}^{n,d}$ with respect to composition with elements in $\mathcal{F}^{d,d}_{\mathrm{icx}}$, as given in (33). This hypothesis will enable us to encompass a wide range of theoretical findings described in [5].

Assumption 1. For any $g \in \mathcal{F}^{n,d}$ and $h \in \mathcal{F}^{d,d}_{\mathrm{icx}}$,

$$h \circ g \in \mathcal{F}^{n,d}.$$

Remark 2.1. Let $\mathcal{F}^{n,d}$ be a class of measurable bounded functions from $\mathbb{R}^n$ to $\mathbb{R}^d$ which fulfils Assumption 1. Then, for any $X, Y \in L^\infty_n(\Omega)$, the following implication holds:

$$X \leq_{\mathcal{F}^{n,d}} Y \quad \Longrightarrow \quad \forall\, g \in \mathcal{F}^{n,d},\ g(X) \leq_{\mathcal{F}^{d,d}_{\mathrm{icx}}} g(Y).$$

In order to verify this claim, let $X, Y \in L^\infty_n(\Omega)$ such that $X \leq_{\mathcal{F}^{n,d}} Y$. According to Definition 2.1.**i**, we aim to show that, for any $g \in \mathcal{F}^{n,d}$ and $h \in \mathcal{F}^{d,d}_{\mathrm{icx}}$,

$$\mathbf{E}^+\bigl[h\bigl(g(Y)\bigr)\bigr] \subseteq \mathbf{E}^+\bigl[h\bigl(g(X)\bigr)\bigr].$$

Due to Assumption 1, whenever $g \in \mathcal{F}^{n,d}$ and $h \in \mathcal{F}^{d,d}_{\mathrm{icx}}$, it is ensured that $h \circ g \in \mathcal{F}^{n,d}$ as well. Then, by Definition 2.1.**ii**,

$$\mathbf{E}^+\bigl[h\bigl(g(Y)\bigr)\bigr] \equiv \mathbf{E}^+\bigl[(h\circ g)(Y)\bigr] \subseteq \mathbf{E}^+\bigl[(h\circ g)(X)\bigr] \equiv \mathbf{E}^+\bigl[h\bigl(g(X)\bigr)\bigr].$$

As we proceed in our discussion, we refer to any (measurable) function $g\colon \mathbb{R}^n \to \mathbb{R}^d$, possibly with respect to K or $\widetilde{K}$, as: $\bigl(\widetilde{K},K\bigr)$-*increasing* if, for any $x, y \in \mathbb{R}^n$,

$$x \leq_{\widetilde{K}} y \quad \Longrightarrow \quad g(x) \leq_K g(y);$$

$\bigl(\widetilde{K},K\bigr)$-*decreasing* if, for any $x, y \in \mathbb{R}^n$,

$$x \leq_{\widetilde{K}} y \quad \Longrightarrow \quad g(y) \leq_K g(x);$$

K-*convex* if, for any $x, y \in \mathbb{R}^n$ and $\lambda \in]0,1[$,

$$g\bigl(\lambda x + (1-\lambda)y\bigr) \leq_K \lambda g(x) + (1-\lambda) g(y);$$

symmetric if, for any $x \in \mathbb{R}^n$ and $\sigma \in S_n$ (with S_n representing the symmetric group of order n),

$$g(x_\sigma) = g(x),$$

where

$$x_\sigma \coloneqq \bigl[x_{\sigma(1)}, \ldots, x_{\sigma(n)}\bigr]^{\mathsf{T}}. \tag{35}$$

An additional order that we seek to examine in this paper is the $\prec$-*Schur convex order*. For this purpose, we revisit the concepts of *strong* and *weak Schur-orderings* in $\mathbb{R}^n$, along with some of their equivalent characterizations. Further details can be found, among others, in [37] and [33].

Definition 2.2. For any $x = [x_1, \dots, x_n]^\mathsf{T}, y = [y_1, \dots, y_n]^\mathsf{T} \in \mathbb{R}^n$, let

$$x_{(1)} \geq \cdots \geq x_{(n)}, \quad y_{(1)} \geq \cdots \geq y_{(n)}$$

stand for the n components of x and y rearranged in *decreasing* order, respectively. Then:

$$x \prec y$$

(*strong Schur-order*) if, and only if, for any $k = 1, \dots, n-1$,

$$\begin{cases} \sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)}, \\ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i; \end{cases}$$

alternatively,

$$x \preceq y$$

(*weak Schur-order*) if, and only if, for any $k = 1, \dots, n$,

$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)}.$$

Consistent with Definition 2.2, we say that a function $g\colon \mathbb{R}^n \to \mathbb{R}^d$ is: $\prec$-, or *strongly-Schur*, *K-convex* if, for any $x, y \in \mathbb{R}^n$,

$$x \prec y \quad \Longrightarrow \quad g(y) \leq_K g(x);$$

$\preceq$-, or *weakly-Schur*, *K-convex* if, for any $x, y \in \mathbb{R}^n$,

$$x \preceq y \quad \Longrightarrow \quad g(y) \leq_K g(x).$$

In light of Definition 2.2, it is evident that, for any $x, y \in \mathbb{R}^n$, if $x \prec y$, then $x \preceq y$. In particular, for functions mapping from $\mathbb{R}^n$ to $\mathbb{R}^d$, $\preceq$-Schur K-convexity implies $\prec$-Schur K-convexity.

The appeal of Schur-convex functions lies in their ability to formalize intuitive comparisons within structured datasets. In economic theory, they provide a rigorous framework for examining income inequality, as their monotonicity under majorization aligns with widely accepted principles of social welfare. In statistics, they shape the behavior of symmetric statistics, yielding meaningful inequalities and enhanced insights into distributional properties. In mathematical optimization, they offer a systematic methodology for addressing problems that involve symmetry and equitable resource allocation.

These broad implications underscore both their theoretical significance and practical utility. However, despite their well-established use in scalar-valued settings, many contemporary problems require a more adaptable approach capable of handling vector-valued mappings. This need is particularly evident in risk theory, where multivariate risk measures often necessitate scrupulous assessments of multidimensional distributions.

Example 2.1. We provide a variety of basic examples illustrating strongly-Schur and weakly-Schur convex functions here (refer back to Definition 2.2).

I. Let $U \subseteq \mathbb{R}^n$, and consider the following two subsets of $\mathbb{R}^n$ corresponding to it:

$$U_{+,\prec} := \big\{ y \in \mathbb{R}^n \,\big|\, \exists\, x \in U : x \prec y \big\}, \quad U_{+,\preceq} := \big\{ y \in \mathbb{R}^n \,\big|\, \exists\, x \in U : x \preceq y \big\}. \tag{36}$$

In particular, by the reflexivity of $\prec$, it holds that

$$U \subseteq U_{+,\prec} \subseteq U_{+,\preceq}.$$

Therefore, the two indicator functions

$$\mathbb{1}_{U_{+,\prec}}, \quad \mathbb{1}_{U_{+,\preceq}}$$

are $\prec$-Schur convex, respectively, $\preceq$-Schur convex (and thus $\prec$-Schur convex), as functions from $\mathbb{R}^n$ to $\{0,1\} \subseteq \mathbb{R}$. Indeed, for any $a, b \in \mathbb{R}^n$, if $a \prec b$ (respectively, $a \preceq b$), then, by (36), if $a \in U_{+,\prec}$ (respectively, $a \in U_{+,\preceq}$), then $b \in U_{+,\prec}$ (respectively, $b \in U_{+,\preceq}$) as well, because both $\prec$ and $\preceq$ are transitive. Consequently, we conclude that $\mathbb{1}_{U_{+,\prec}}(a) \leq \mathbb{1}_{U_{+,\prec}}(b)$ (respectively, $\mathbb{1}_{U_{+,\preceq}}(a) \leq \mathbb{1}_{U_{+,\preceq}}(b)$).

II. For any bounded function $g\colon \mathbb{R}^n \to \mathbb{R}^d$ that is $\prec$-Schur K-convex (respectively, $\preceq$-Schur K-convex), and for $w \in K^+$, the scalar function $g_w\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$g_w(x) \coloneqq w^\mathsf{T} g(x), \quad x \in \mathbb{R}^n$$

is $\prec$-Schur convex (respectively, $\preceq$-Schur convex). Indeed, for any $x, y \in \mathbb{R}^n$, if $x \prec y$ (respectively, $x \preceq y$), then $g(y) - g(x) \in K$, and from (30), we obtain $w^\mathsf{T}\big(g(y) - g(x)\big) \geq 0$, i.e.,

$$g_w(x) \equiv w^\mathsf{T} g(x) \leq w^\mathsf{T} g(y) \equiv g_w(y).$$

Lastly, we offer examples of function classes of type $\mathcal{F}^{n,d}$ that align with the aforementioned properties and, as can be readily checked, satisfy Assumption 1 when coupled with $\mathcal{F}^{d,d}_{\mathrm{icx}}$ (see (33)). These classes will prove to be particularly well-suited to the objectives of our analysis.

$$\mathcal{F}^{n,d}_{\mathrm{inc}} \equiv \mathcal{F}^{n,d}_{\mathrm{inc}}\big(\widetilde{K}, K\big) \coloneqq \Big\{ g\colon \mathbb{R}^n \to \mathbb{R}^d \;\Big|\; g \text{ is bounded and } \big(\widetilde{K}, K\big)\text{-increasing} \Big\}.$$

$$\mathcal{F}^{n,d}_{\mathrm{dec}} \equiv \mathcal{F}^{n,d}_{\mathrm{dec}}\big(\widetilde{K}, K\big) \coloneqq \Big\{ g\colon \mathbb{R}^n \to \mathbb{R}^d \;\Big|\; g \text{ is bounded and } \big(\widetilde{K}, K\big)\text{-decreasing} \Big\}.$$

$$\mathcal{F}^{n,d}_{\mathrm{cx}} \equiv \mathcal{F}^{n,d}_{\mathrm{cx}}(K) \coloneqq \big\{ g\colon \mathbb{R}^n \to \mathbb{R}^d \;\big|\; g \text{ is bounded and } K\text{-convex} \big\}.$$

$$\mathcal{F}^{n,d}_{\mathrm{icx}} \equiv \mathcal{F}^{n,d}_{\mathrm{icx}}\big(\widetilde{K}, K\big) \coloneqq \mathcal{F}^{n,d}_{\mathrm{inc}} \cap \mathcal{F}^{n,d}_{\mathrm{cx}}.$$

$$\mathcal{F}^{n,d}_{\mathrm{decx}} \equiv \mathcal{F}^{n,d}_{\mathrm{decx}}\big(\widetilde{K}, K\big) \coloneqq \mathcal{F}^{n,d}_{\mathrm{dec}} \cap \mathcal{F}^{n,d}_{\mathrm{cx}}.$$

$$\mathcal{F}^{n,d}_{\prec} \equiv \mathcal{F}^{n,d}_{\prec}(K) \coloneqq \big\{ g\colon \mathbb{R}^n \to \mathbb{R}^d \;\big|\; g \text{ is bounded and } \prec\text{-Schur } K\text{-convex} \big\}.$$

$$\mathcal{F}^{n,d}_{\preceq} \equiv \mathcal{F}^{n,d}_{\preceq}(K) \coloneqq \big\{ g\colon \mathbb{R}^n \to \mathbb{R}^d \;\big|\; g \text{ is bounded and } \preceq\text{-Schur } K\text{-convex} \big\}.$$

$$\mathcal{F}^{n,d}_{\mathrm{sym}} \coloneqq \big\{ g\colon \mathbb{R}^n \to \mathbb{R}^d \;\big|\; g \text{ is bounded and symmetric} \big\}.$$

$$\mathcal{F}^{n,d}_{\mathrm{scx}} \equiv \mathcal{F}^{n,d}_{\mathrm{scx}}(K) \coloneqq \mathcal{F}^{n,d}_{\mathrm{sym}} \cap \mathcal{F}^{n,d}_{\mathrm{cx}}.$$

$$\mathcal{F}^{n,d}_{\mathrm{iscx}} \equiv \mathcal{F}^{n,d}_{\mathrm{iscx}}\big(\widetilde{K}, K\big) \coloneqq \mathcal{F}^{n,d}_{\mathrm{inc}} \cap \mathcal{F}^{n,d}_{\mathrm{scx}}.$$

Remark 2.2. We furnish a few interesting observations in this regard, which set the stage for our discussion on consistency with respect to certain stochastic orderings, as outlined above.

I. The set inclusion

$$\mathcal{F}^{n,d}_{\text{scx}} \subseteq \mathcal{F}^{n,d}_{\prec} \tag{37}$$

is satisfied and, correspondingly, the following implication between order relations arises:

$$\leq_{\mathcal{F}^{n,d}_{\prec}} \implies \leq_{\mathcal{F}^{n,d}_{\text{scx}}}$$

(see also Definition 2.2). Indeed, for any $g \in \mathcal{F}^{n,d}_{\text{scx}}$ (i.e., a bounded, symmetric, and convex function) and $x, y \in \mathbb{R}^n$ with $x \prec y$, if, as described in [33], $\{\lambda_\sigma\}_{\sigma \in S_n} \subseteq [0,1]$ is such that

$$\sum_{\sigma \in S_n} \lambda_\sigma = 1, \quad x = \sum_{\sigma \in S_n} \lambda_\sigma y_\sigma$$

where $y_\sigma \coloneqq [y_{\sigma(1)}, \ldots, y_{\sigma(n)}]^\mathsf{T}$ (see (35)), then, clearly,

$$g(x) = g\left(\sum_{\sigma \in S_n} \lambda_\sigma y_\sigma\right) \leq_K \sum_{\sigma \in S_n} \lambda_\sigma g(y_\sigma) = g(y).$$

Thus, $g \in \mathcal{F}^{n,d}_{\prec}$.

II. Under the extra assumption

$$\mathbb{R}^n_+ \doteq [0, \infty[^n \subseteq \widetilde{K},$$

(commonly accepted within our framework), the set inclusion

$$\mathcal{F}^{n,d}_{\text{iscx}} \subseteq \mathcal{F}^{n,d}_{\preceq} \tag{38}$$

holds and, therefore, the implication

$$\leq_{\mathcal{F}^{n,d}_{\preceq}} \implies \leq_{\mathcal{F}^{n,d}_{\text{iscx}}}$$

is fulfilled. In fact, let $g \in \mathcal{F}^{n,d}_{\text{iscx}}$ (i.e., bounded, increasing, symmetric, and convex function) and $x, y \in \mathbb{R}^n$ with $x \preceq y$. Then, one can easily find $z \equiv z_{x,y} \in \mathbb{R}^n$ such that

$$x \prec z \leq_{\mathbb{R}^n_+} y.$$

Since $\mathbb{R}^n_+ \subseteq \widetilde{K}$, it is guaranteed that $\leq_{\mathbb{R}^n_+} \Rightarrow \leq_{\widetilde{K}}$ and, in particular, the second inequality above implies $z \leq_{\widetilde{K}} y$. Hence, also by (37), we arrive at

$$g(x) \leq_K g(z) \leq_K g(y),$$

thereby $g \in \mathcal{F}^{n,d}_{\preceq}$.

We are ready to establish the consistency of set-valued maps of the $\Psi_A(\cdot)$ type, as introduced in (29), under our convex-type stochastic orderings (see Definition 2.1 in particular). To move forward, we impose the following *consistency* assumption regarding each Ψ_α, $\alpha \in A$.

Assumption 2. For all $\alpha \in A$, $\Psi_\alpha(\cdot)$ is $\leq_{\mathcal{F}^{d,d}_{\text{icx}}}$*-consistent* (relatively to the partial order $\subseteq$ on $\mathbb{R}^d$): for any $U, V \in L^\infty_d(\Omega)$,

$$U \leq_{\mathcal{F}^{d,d}_{\text{icx}}} V \implies \Psi_\alpha(V) \subseteq \Psi_\alpha(U)$$

(see Definition 2.1.**i**, and (33)).

Therefore, we share our findings on consistency, which will finalize the current subsection.

Proposition 2.1. *Let $\mathcal{F}^{n,d}$ denote a class of measurable bounded functions from $\mathbb{R}^n$ to $\mathbb{R}^d$. Under Assumption 1 on $\mathcal{F}^{n,d}$ and Assumption 2 on Ψ_α, $\alpha \in A$, the following two statements hold.*

1. *If* $\{g_\alpha\}_{\alpha\in A} \subseteq \mathcal{F}^{n,d}$*, then* $\Psi_A(\cdot)$ *is* $\leq_{\mathcal{F}^{n,d}}$*-consistent: for any* $X, Y \in L_n^\infty(\Omega)$*,*

$$X \leq_{\mathcal{F}^{n,d}} Y \quad \Longrightarrow \quad \Psi_A(Y) \subseteq \Psi_A(X).$$

2. *If* $\mathbb{R}_+^n \subseteq \widetilde{K}$ *and* $\{g_\alpha\}_{\alpha\in A} \subseteq \mathcal{F}_{\mathrm{iscx}}^{n,d}$*, then* $\Psi_A(\cdot)$ *is* $\leq_{\mathcal{F}_{\preceq}^{n,d}}$*-consistent: for any* $X, Y \in L_n^\infty(\Omega)$*,*

$$X \leq_{\mathcal{F}_{\preceq}^{n,d}} Y \quad \Longrightarrow \quad \Psi_A(Y) \subseteq \Psi_A(X).$$

Proof. Each of the two statements will be addressed individually, relying on assumptions and remarks stated up to this point.

1. Let $\{g_\alpha\}_{\alpha\in A} \subseteq \mathcal{F}^{n,d}$. Then, by Assumption 1 and Remark 2.1, we have that, for all $\alpha \in A$,

$$X \leq_{\mathcal{F}^{n,d}} Y \quad \Longrightarrow \quad g_\alpha(X) \leq_{\mathcal{F}_{\mathrm{icx}}^{d,d}} g_\alpha(Y).$$

Therefore, by Assumption 2, it holds that, for all $\alpha \in A$,

$$X \leq_{\mathcal{F}^{n,d}} Y \quad \Longrightarrow \quad \Psi_\alpha(g_\alpha(Y)) \subseteq \Psi_\alpha(g_\alpha(X)).$$

The argument is now complete, following directly from the definition of $\Psi_A(\cdot)$ (see (29)).

2. Let $\mathbb{R}_+^n \subseteq \widetilde{K}$ and $\{g_\alpha\}_{\alpha\in A} \subseteq \mathcal{F}_{\mathrm{iscx}}^{n,d}$. Then, by (38) in Remark 2.2 (point **II**), and again by Assumption 1 and Remark 2.1, we derive that, for all $\alpha \in A$,

$$X \leq_{\mathcal{F}_{\preceq}^{n,d}} Y \quad \Longrightarrow \quad X \leq_{\mathcal{F}_{\mathrm{iscx}}^{n,d}} Y \quad \Longrightarrow \quad g_\alpha(X) \leq_{\mathcal{F}_{\mathrm{icx}}^{d,d}} g_\alpha(Y)$$

and the conclusion can be inferred as in point **1**. □

2.2 On Schur-convex stochastic ordering

In this subsection, we delve into an equivalent description of a *strengthened form* of the $\prec$-Schur K-convex stochastic ordering

$$\leq_{\mathcal{F}_{\prec}^{n,d}},$$

previously introduced and discussed in Subsection 2.1 (alongside related elements). Specifically, we begin by presenting an enhanced variant of Definition 2.1.**ii**: for any $X, Y \in L_n^\infty(\Omega)$,

$$\begin{aligned} X \leq_{\mathcal{F}^{n,d}}^{s} Y \quad \overset{\mathbf{def}}{\Longleftrightarrow} \quad & \forall\, g \in \mathcal{F}^{n,d},\ \forall\, w \in K^+ \setminus \{0\},\ \forall\, \mathbf{Q} \equiv \mathbf{Q}_w \in \mathcal{Q}(w), \\ & \mathbf{E}_{\mathbf{Q},w}^{+}\big[g(Y)\big] \subseteq \mathbf{E}_{\mathbf{Q},w}^{+}\big[g(X)\big] \end{aligned} \tag{39}$$

(refer to (30), (31) and (32)). Definition (39) can also be formulated through the equivalence below:

$$\begin{aligned} (39) \quad \Longleftrightarrow \quad & \forall\, g \in \mathcal{F}^{n,d},\ \forall\, w \in K^+ \setminus \{0\},\ \forall\, \mathbf{Q} \in \mathcal{Q}(w), \\ & w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(X)\big] \leq w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(Y)\big] \end{aligned} \tag{40}$$

(with $X, Y \in L_n^\infty(\Omega)$). Indeed, by means of (32), it is straightforward to verify that (39) is equivalent to the following implication: for any $g \in \mathcal{F}^{n,d}$, $w \in K^+ \setminus \{0\}$, $\mathbf{Q} \in \mathcal{Q}(w)$ and $z \in \mathbb{R}^d$,

$$w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(Y)\big] \leq w^\mathsf{T} z \quad \Longrightarrow \quad w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(X)\big] \leq w^\mathsf{T} z.$$

This fact immediately leads us to (40).

Remark 2.3. For any $X, Y \in L_n^\infty(\Omega)$, the subsequent implication holds:

$$X \prec Y \ (\mathbf{P}\text{-a.s.}) \quad \Longrightarrow \quad X \leq^s_{\mathcal{F}^{n,d}_{\prec}} Y.$$

This claim is valid since, if $X \prec Y$ ($\mathbf{P}$-a.s.), then, for any $g \in \mathcal{F}^{n,d}_{\prec}$, $g(Y) - g(X) \in K$ ($\mathbf{P}$-a.s.). Thus, by (30) and (31), we get that, for any $w \in K^+ \setminus \{0\}$ and $\mathbf{Q} \in \mathcal{Q}(w)$,

$$w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(Y) - g(X)\big] \equiv \sum_{i=1}^{d} w_i\, \mathbf{E}^{\mathbf{Q}_i}\big[g(Y) - g(X)\big] = \mathbf{E}^{\mathbf{P}}\left[\sum_{i=1}^{d} w_i \frac{d\mathbf{Q}_i}{d\mathbf{P}}\Big(g(Y) - g(X)\Big)\right] \geq 0,$$

which, in accordance with (40), signifies that $X \leq^s_{\mathcal{F}^{n,d}_{\prec}} Y$.

Next, the assumption regarding K^+, outlined and applied below, is required (see (30)).

Assumption 3. $\mathbb{R}^d_+ \subsetneq K$, that is,

$$K^+ \subsetneq \mathbb{R}^d_+$$

and, for any $w \in K^+ \setminus \{0\}$ and $i = 1, \ldots, d$,

$$w_i \neq 0.$$

Remark 2.4. Suppose that Assumption 3 is satisfied. For any $X, Y \in L_n^\infty(\Omega)$, if $X \leq^s_{\mathcal{F}^{n,d}_{\prec}} Y$, then, for any $\varphi \in \mathcal{F}^{n,1}_{\prec}$ (classical $\prec$-Schur convex function), $w \in K^+ \setminus \{0\}$, $\mathbf{Q} \in \mathcal{Q}(w)$ and $j = 1, \ldots, d$,

$$\mathbf{E}^{\mathbf{Q}_j}\big[\varphi(X)\big] \leq \mathbf{E}^{\mathbf{Q}_j}\big[\varphi(Y)\big]. \tag{41}$$

We now proceed to prove the above statement. By virtue of (40), for any $X, Y \in L_n^\infty(\Omega)$, if $X \leq^s_{\mathcal{F}^{n,d}_{\prec}} Y$, then, for any $g \in \mathcal{F}^{n,d}_{\prec}$, $w \in K^+ \setminus \{0\}$ and $\mathbf{Q} \in \mathcal{Q}(w)$,

$$w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(X)\big] \leq w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(Y)\big]. \tag{42}$$

As a consequence, given $\varphi \in \mathcal{F}^{n,1}_{\prec}$ and $j = 1, \ldots, d$, if we consider, among all possible choices, the function $g \equiv g_{\varphi,j} = [g_1, \ldots, g_d]^\mathsf{T} \colon \mathbb{R}^n \to \mathbb{R}^d$ determined, for any $i = 1, \ldots, d$, by

$$g_i = \begin{cases} \varphi, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

then $g \in \mathcal{F}^{n,d}_{\prec}$ (since $\varphi \in \mathcal{F}^{n,1}_{\prec}$ and $\mathbb{R}^d_+ \subseteq K$), and therefore, due to (42),

$$\mathbf{E}^{\mathbf{Q}_j}\big[w_j\,\varphi(X)\big] \leq \mathbf{E}^{\mathbf{Q}_j}\big[w_j\,\varphi(Y)\big],$$

whence (41) follows (by canceling $w_j > 0$).

The upcoming result offers a characterization of the ordering relation

$$\leq^s_{\mathcal{F}^{n,d}_{\prec}}$$

by extending Theorem 3 in [33] to our set-valued setting. For a more in-depth understanding of the topic, please refer to [39] and [38] as well.

Theorem 2.1. *Let Assumption 3 hold and consider*

$$\mathcal{F}^{n,d} = \mathcal{F}^{n,d}_{\prec}.$$

Then, for any $X, Y \in L_n^\infty(\Omega)$, the following three conditions are equivalent.

1. $X \leq^{\mathrm{s}}_{\mathcal{F}^{n,d}_{\prec}} Y$ *(see* (39) *or* (40)*).*

2. *For any $w \in K^+ \setminus \{0\}$, $\mathbf{Q} \in \mathcal{Q}(w)$ and $i = 1, \dots, d$, there exists a Markov kernel*

$$T_i(\cdot\,, \text{-})$$

on $\big(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\big)$, depending on w, $\mathbf{Q}_i$, X and Y, satisfying the following two constraints.

2a. $(\mathbf{Q}_i)^Y = T_i\,(\mathbf{Q}_i)^X$*, that is, for any measurable bounded function $\upsilon\colon \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} \upsilon(y)\,(\mathbf{Q}_i)^Y(dy) = \int_{\mathbb{R}^n} \left[\, \int_{\mathbb{R}^n} \upsilon(y)\, T_i(x,dy) \right] (\mathbf{Q}_i)^X(dx). \tag{43}$$

2b. *For any $g \in \mathcal{F}^{n,d}_{\prec}$,*

$$\sum_{j=1}^{d} \int_{\mathbb{R}^n} w_j\, g_j(x)\,(\mathbf{Q}_j)^X(dx) \le \sum_{j=1}^{d} \int_{\mathbb{R}^n} \left[\, \int_{\mathbb{R}^n} w_j\, g_j(y)\, T_j(x,dy) \right] (\mathbf{Q}_j)^X(dx). \tag{44}$$

3. *There exist a measurable space $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}\big)$ and two $\mathbb{R}^n$-valued random vectors $\widetilde{X}, \widetilde{Y}$ on $\widetilde{\Omega}$ such that, for any $w \in K^+ \setminus \{0\}$, $\mathbf{Q} \in \mathcal{Q}(w)$ and $i = 1, \dots, d$, there exists a complete probability measure $\widetilde{\mathbf{Q}}_i$ on $\widetilde{\mathcal{F}}$, depending on w, $\mathbf{Q}_i$, X and Y, meeting the following three properties.*

3a. *It holds that*

$$\big(\widetilde{\mathbf{Q}}_i\big)^{\widetilde{X}} = (\mathbf{Q}_i)^X, \quad \big(\widetilde{\mathbf{Q}}_i\big)^{\widetilde{Y}} = (\mathbf{Q}_i)^Y. \tag{45}$$

In particular, $\widetilde{X}, \widetilde{Y} \in L^\infty_n\big(\widetilde{\Omega}\big)$.

3b. $\widetilde{\mathbf{Q}}_i$*-a.s.,*

$$\widetilde{X} \prec \widetilde{Y}.$$

3c. *For any $g \in \mathcal{F}^{n,d}_{\prec}$,*

$$w^{\mathsf{T}}\, \mathbf{E}^{\widetilde{\mathbf{Q}}_i}\big[g\big(\widetilde{X}\big)\big] \le w^{\mathsf{T}}\, \mathbf{E}^{\widetilde{\mathbf{Q}}_i}\big[g\big(\widetilde{Y}\big)\big]. \tag{46}$$

Proof. To demonstrate the equivalence in question, we will prove the chain of implications:

$$\mathbf{1} \quad \Rightarrow \quad \mathbf{3} \quad \Rightarrow \quad \mathbf{2} \quad \Rightarrow \quad \mathbf{1}.$$

1 $\Rightarrow$ **3**. Fix a nonempty open set $U \subseteq \mathbb{R}^n$, and let

$$U_{+,\prec} = \big\{ y \in \mathbb{R}^n \;\big|\; \exists\, x \in U : x \prec y \big\},$$

as defined in (36) (Example 2.1, point **I**). Then, $\varphi \coloneqq \mathbb{1}_{U_{+,\prec}} \in \mathcal{F}^{n,1}_{\prec}$, and owing to Remark 2.4—in particular, (41)—we obtain that, for any $w \in K^+ \setminus \{0\}$, $\mathbf{Q} \in \mathcal{Q}(w)$ and $i = 1, \dots, d$,

$$(\mathbf{Q}_i)^X[U_{+,\prec}] \le (\mathbf{Q}_i)^Y[U_{+,\prec}]. \tag{47}$$

In alignment with Theorem 3 in [33], we also set

$$\omega \coloneqq \big\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \;\big|\; x \prec y \big\}.$$

One can readily verify that ω is a closed subset of $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore, taking into account the two canonical projections $\pi_1, \pi_2\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given, for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, by

$$\begin{cases} \pi_1(x, y) \coloneqq x, \\ \pi_2(x, y) \coloneqq y, \end{cases}$$

it follows that

$$U \subseteq U_{+,\prec} = \pi_2\big(\omega \cap (U \times \mathbb{R}^n)\big)$$

and thus, from (47), we attain that, for any $i = 1, \ldots, d$,

$$(\mathbf{Q}_i)^X[U] \le (\mathbf{Q}_i)^Y\big[\pi_2\big(\omega \cap (U \times \mathbb{R}^n)\big)\big].$$

Consequently, by the fundamental Theorem 11 in [39], a probability measure $\widetilde{\mathbf{Q}}_i$ is assured to exist, for each $i = 1, \ldots, d$, on the measurable (Borel) space

$$\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}\big) := \big(\mathbb{R}^{2n}, \mathcal{B}(\mathbb{R}^{2n})\big),$$

exhibiting the subsequent peculiarities: $(\widetilde{\mathbf{Q}}_i)^{\pi_1} = (\mathbf{Q}_i)^X$, $(\widetilde{\mathbf{Q}}_i)^{\pi_2} = (\mathbf{Q}_i)^Y$, and

$$\widetilde{\mathbf{Q}}_i(\omega) \equiv \widetilde{\mathbf{Q}}_i[\pi_1 \prec \pi_2] = 1.$$

To conclude this opening part of the proof, it is therefore sufficient to define

$$\widetilde{X} := \pi_1, \quad \widetilde{Y} := \pi_2.$$

Observe that (46) arises from (40) and (45), since $X \le^s_{\mathcal{F}^{n,d}_\prec} Y$ is initially assumed.

3 $\Rightarrow$ **2**. For any $w \in K^+ \setminus \{0\}$, $\mathbf{Q} \in \mathcal{Q}(w)$ and $i = 1, \ldots, d$, let T_i denote the regular conditional distribution of $\widetilde{Y}$ given $\widetilde{X}$ w.r.t. $\widetilde{\mathbf{Q}}_i$ (refer to Remark 2.5 following the proof, if needed). Then, (43) directly derives from (45). On the other hand, with respect to (44), we have that, for any $g \in \mathcal{F}^{n,d}_\prec$,

$$\begin{aligned}
\sum_{i=1}^d \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} w_i\, g_i(y)\, T_i(x,dy)\right] (\mathbf{Q}_i)^X(dx) &= \sum_{i=1}^d \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} w_i\, g_i(y)\, T_i(x,dy)\right] \big(\widetilde{\mathbf{Q}}_i\big)^{\widetilde{X}}(dx) \\
&= \sum_{i=1}^d \int_{\mathbb{R}^n} w_i\, g_i(y)\, \big(\widetilde{\mathbf{Q}}_i\big)^{\widetilde{Y}}(dy) \\
&\equiv w^\mathsf{T}\, \mathbf{E}^{\widetilde{\mathbf{Q}}_i}\big[g\big(\widetilde{Y}\big)\big] \\
&\ge w^\mathsf{T}\, \mathbf{E}^{\widetilde{\mathbf{Q}}_i}\big[g\big(\widetilde{X}\big)\big] \\
&\equiv \sum_{i=1}^d \int_{\mathbb{R}^n} w_i\, g_i(x)\, \big(\widetilde{\mathbf{Q}}_i\big)^{\widetilde{X}}(dx) \\
&= \sum_{i=1}^d \int_{\mathbb{R}^n} w_i\, g_i(x)\, (\mathbf{Q}_i)^X(dx)
\end{aligned}$$

(in the preceding steps, the inequality in the fourth line corresponds to (46)).

2 $\Rightarrow$ **1**. For any $g \in \mathcal{F}^{n,d}_\prec$, $w \in K^+ \setminus \{0\}$ and $\mathbf{Q} \in \mathcal{Q}(w)$, leveraging (44) and (43), we can write

$$\begin{aligned}
w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(X)\big] &\equiv \sum_{i=1}^d \int_{\mathbb{R}^n} w_i\, g_i(x)\, (\mathbf{Q}_i)^X(dx) \\
&\le \sum_{i=1}^d \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} w_i\, g_i(y)\, T_i(x,dy)\right] (\mathbf{Q}_i)^X(dx) \\
&= \sum_{i=1}^d \int_{\mathbb{R}^n} w_i\, g_i(y)\, (\mathbf{Q}_i)^Y(dy) \\
&\equiv w^\mathsf{T}\, \mathbf{E}^{\mathbf{Q}}\big[g(Y)\big],
\end{aligned}$$

that is, (40). The proof of the current sub-case, and thus of our theorem, is now complete. □

Remark 2.5. With regard to point **2** of Theorem 2.1, it is worth noting that any *Polish space* E (a separable and completely metrizable topological space, such as any Euclidean space), equipped with its Borel σ-algebra—i.e., $\big(E, \mathcal{B}(E)\big)$—is *regular* with respect to the notion of regular conditional distributions/probabilities. Specifically, for any probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, and any random variables $Y' \colon (\Omega', \mathcal{F}') \to \big(E, \mathcal{B}(E)\big)$ and X' on $(\Omega', \mathcal{F}')$ with values in an arbitrary measurable set, there exists the regular conditional distribution of Y' given X' w.r.t. $\mathbf{P}'$.

3 Law-invariant convex risk measures

In the present and concluding section, we introduce a noteworthy family of set-valued maps

$$\Psi_\alpha \colon L_d^\infty(\Omega) \to \mathcal{P}\big(\mathbb{R}^d\big), \quad \alpha \in A$$

(where $\mathcal{P}\big(\mathbb{R}^d\big)$ indicates the power set of $\mathbb{R}^d$), that satisfy Assumption 2, as outlined in Section 2. Based on Proposition 2.1, we aim to establish that this condition is nearly sufficient for constructing a convexly consistent set-valued function of the form $\Psi_A(\cdot)$, as (29) specifies, which may be linked to a maximal correlation set-valued risk measure for portfolio aggregation (see (26), Proposition 1.3, and (28), in Section 1), as highlighted in the introduction to Section 2.

To this end, let

$$R_{\mathcal{H}} \colon L_d^\infty(\Omega) \to \mathbb{F}_M$$

stands for a *law-invariant* convex set-valued risk measure associated with K (refer to Section 1). This conveys that $R_{\mathcal{H}}(\cdot)$ assigns the same (set-)value to any random vector in $L_d^\infty(\Omega)$ sharing the same joint distribution (w.r.t. $\mathbf{P}$): for any $U, \tilde{U} \in L_d^\infty(\Omega)$,

$$\tilde{U} \sim U \quad \Longrightarrow \quad R_{\mathcal{H}}\big(\tilde{U}\big) = R_{\mathcal{H}}(U). \tag{48}$$

Let us also assume the existence of a set

$$\mathcal{W} \equiv \mathcal{W}_K^{\mathbf{P}} \subseteq \big(K^+ \setminus \{0\}\big) \times \big(\mathcal{M}_1^{\mathbf{P}}\big)^d$$

in correspondence with which a *dual representation* formula for $R_{\mathcal{H}}(\cdot)$, à la [21], holds. Specifically, we posit that, for any $U = [U_1, \ldots, U_d]^\mathsf{T} \in L_d^\infty(\Omega)$,

$$R_{\mathcal{H}}(U) = \bigcap_{(w,\mathbf{Q}) \in \mathcal{W}} \Big\{ -\alpha(w, \mathbf{Q}) + R_{(w,\mathbf{Q})}(U) \Big\} \tag{49}$$

where, if we denote

$$\mathcal{A}_{R_{\mathcal{H}}} := \Big\{ V = [V_1, \ldots, V_d]^\mathsf{T} \in L_d^\infty(\Omega) \;\Big|\; 0 \in R_{\mathcal{H}}(V) \Big\},$$

then, for any $(w, \mathbf{Q}) \in \mathcal{W}$ with $w = [w_1, \ldots, w_d]^\mathsf{T}$ and $\mathbf{Q} = [\mathbf{Q}_1, \ldots, \mathbf{Q}_d]^\mathsf{T}$,

$$-\alpha(w, \mathbf{Q}) := \left\{ z = [z_1, \ldots, z_d]^\mathsf{T} \in M \;\middle|\; \inf_{V \in \mathcal{A}_{R_{\mathcal{H}}}} \sum_{i=1}^d w_i \mathbf{E}\left[V_i \frac{d\mathbf{Q}_i}{d\mathbf{P}} \right] \le \sum_{i=1}^d w_i z_i \right\}$$

and

$$R_{(w,\mathbf{Q})}(U) := \left\{ z \in M \;\middle|\; \inf_{\tilde{U} \sim U} \sum_{i=1}^d w_i \mathbf{E}\left[\tilde{U}_i \frac{d\mathbf{Q}_i}{d\mathbf{P}} \right] + \sum_{i=1}^d w_i z_i \ge 0 \right\}. \tag{50}$$

Prior to our *consistency* findings concerning $R_{\mathcal{H}}(\cdot)$, we lay out the following statement as a set-valued extension of a vectorial result reported in Corollary 4.59, Chapter 4, of [17]. Beyond its potential application, this result remains valuable on its own, both in terms of the formulation and the proof. We also encourage comparison with Lemma 2.3 in [36] (for related insights, see [15]).

Lemma 3.1. *Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra on Ω. Then, for any $U = [U_1, \ldots, U_d]^\mathsf{T} \in L_d^\infty(\Omega)$,*

$$R_\mathcal{H}(U) \subseteq R_\mathcal{H}\big(\mathbf{E}[U\,|\,\mathcal{G}]\big), \tag{51}$$

where

$$\mathbf{E}[U\,|\,\mathcal{G}] \coloneqq \big[\mathbf{E}[U_1|\mathcal{G}], \ldots, \mathbf{E}[U_d|\mathcal{G}]\big]^\mathsf{T}. \tag{52}$$

Proof. Firstly, by (49) and (50), the desired inclusion (51) can be reformulated as:

$$\forall\, (w, \mathbf{Q}) \in \mathcal{W},\ R_{(w,\mathbf{Q})}(U) \subseteq R_{(w,\mathbf{Q})}\big(\mathbf{E}[U|\mathcal{G}]\big). \tag{53}$$

For this reason, fixing an arbitrary $(w, \mathbf{Q}) \in \mathcal{W}$, we focus on the map $R_{(w,\mathbf{Q})}(\cdot)$, as defined in (50).

Drawing from [9, Section 5], we find that

$$\begin{aligned}\inf_{\tilde U \sim U} \sum_{i=1}^d w_i \mathbf{E}\Big[\tilde U_i \frac{d\mathbf{Q}_i}{d\mathbf{P}}\Big] &= \inf_{\tilde U \sim U} \sum_{i=1}^d w_i \inf_{\tilde U_i \sim U_i} \mathbf{E}\Big[\tilde U_i \frac{d\mathbf{Q}_i}{d\mathbf{P}}\Big] \\ &= \inf_{\tilde U \sim U} \sum_{i=1}^d w_i \inf_{\widetilde{\mathbf{Q}}_i \sim \mathbf{Q}_i} \inf_{\tilde U_i \sim U_i} \mathbf{E}\Big[\tilde U_i \frac{d\mathbf{Q}_i}{d\mathbf{P}}\Big]\end{aligned} \tag{54}$$

(the technical issue here lies in the $\leq$ relations). By Lemma 4.55 and the proof of Theorem 4.57 in [17], moreover, for any $i = 1, \ldots, d$ and $\widetilde{\mathbf{Q}}_i \sim \mathbf{Q}_i$ in $\mathcal{M}_1^\mathbf{P}$, there exists a probability measure $\mu_{\widetilde{\mathbf{Q}}_i}$ on $\mathcal{B}\big(]0,1]\big)$, absolutely continuous w.r.t. the Lebesgue measure (i.e., in $\mathcal{M}_1\big(]0,1]\big)$), such that

$$\inf_{\tilde U_i \sim U_i} \mathbf{E}\Big[\tilde U_i \frac{d\mathbf{Q}_i}{d\mathbf{P}}\Big] = -\int_0^1 q_{-U_i}(t)\, q_{\frac{d\mathbf{Q}_i}{d\mathbf{P}}}(t)\, dt = -\int_0^1 \mathrm{AV@R}_\lambda(U_i)\, \mu_{\widetilde{\mathbf{Q}}_i}(d\lambda) \tag{55}$$

(by also employing the standard notions of *quantile* and *Average Value at Risk* at level $\lambda \in\,]0,1]$).

From the combination of (54) and (55), we derive that

$$\inf_{\tilde U \sim U} \sum_{i=1}^d w_i\, \mathbf{E}\Big[\tilde U_i \frac{d\mathbf{Q}_i}{d\mathbf{P}}\Big] = \sum_{i=1}^d w_i \inf_{\mu \in \mathcal{M}_1(]0,1])} \int_0^1 \big[-\mathrm{AV@R}_\lambda(U_i)\big]\, \mu(d\lambda), \tag{56}$$

and therefore, by ((50) and) (56), the inclusion (53) is equivalent to requiring the following condition: for any $w \in K^+\backslash\{0\}$ and $z = [z_1, \ldots, z_d]^\mathsf{T} \in M$, if

$$\sum_{i=1}^d w_i \inf_{\mu \in \mathcal{M}_1(]0,1])} \int_0^1 \big[-\mathrm{AV@R}_\lambda(U_i)\big]\, \mu(d\lambda) + \sum_{i=1}^d w_i z_i \geq 0,$$

then

$$\sum_{i=1}^d w_i \inf_{\mu \in \mathcal{M}_1(]0,1])} \int_0^1 \Big[-\mathrm{AV@R}_\lambda\big(\mathbf{E}[U_i|\mathcal{G}]\big)\Big]\, \mu(d\lambda) + \sum_{i=1}^d w_i z_i \geq 0.$$

To establish this implication, and hence the thesis, we proceed by noting that, for any $\lambda \in\,]0,1]$, $\mathrm{AV@R}_\lambda(\cdot)$ is a law-invariant (scalar) convex risk measure that is continuous from above, and thus, by (equation (3.27) and) Corollary 4.59 in [17], for any $i = 1, \ldots, d$,

$$-\mathrm{AV@R}_\lambda(U_i) \leq -\mathrm{AV@R}_\lambda\big(\mathbf{E}[U_i\,|\,\mathcal{G}]\big),$$

(see [17, Chapter 4] for additional details). From this, it directly follows that, for any $w \in K^+\backslash\{0\}$,

$$\sum_{i=1}^d w_i \inf_{\mu \in \mathcal{M}_1(]0,1])} \int_0^1 \big[-\mathrm{AV@R}_\lambda(U_i)\big]\, \mu(d\lambda) \leq \sum_{i=1}^d w_i \inf_{\mu \in \mathcal{M}_1(]0,1])} \int_0^1 \Big[-\mathrm{AV@R}_\lambda\big(\mathbf{E}[U_i|\mathcal{G}]\big)\Big]\, \mu(d\lambda),$$

and finally, all considered, the proof is concluded. □

Lemma 3.1 may be readily extended to capture a broader notion of monotonicity, as a corollary of the lemma. Further considerations are proposed in Remark 3.1, which follows thereafter.

Corollary 3.1. *Let $\mathcal{G}_1, \mathcal{G}_2 \subseteq \mathcal{F}$ be two σ-algebras on Ω. If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then, for any $U \in L_d^\infty(\Omega)$,*

$$R_{\mathcal{H}}\big(\mathbf{E}[U \,|\, \mathcal{G}_2]\big) \subseteq R_{\mathcal{H}}\big(\mathbf{E}[U \,|\, \mathcal{G}_1]\big). \tag{57}$$

Proof. This outcome is immediately obtained from (51) by applying the well-known tower property of conditional expectation. Specifically, since $\mathcal{G}_1 \subseteq \mathcal{G}_2$, replacing $\mathcal{G}$ with $\mathcal{G}_1$ and U with $\mathbf{E}[U \,|\, \mathcal{G}_2]$ in (51) yields (57) exactly (see also (52)). Thus, the claim holds. □

Remark 3.1. Post facto, Lemma 3.1 and its Corollary 3.1 assert two equivalent propositions. In particular, (57) clearly simplifies to (51) when $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_2 = \mathcal{F}$. Furthermore, regarding solely Lemma 3.1, the choice of the trivial σ-algebra $\mathcal{G} = \{\emptyset, \Omega\}$ on Ω brings about (51) reducing to the following inclusion (which does not involve explicit conditional expectations): for any $U \in L_d^\infty(\Omega)$,

$$R_{\mathcal{H}}(U) \subseteq R_{\mathcal{H}}\big(\mathbf{E}[U]\big).$$

The forthcoming two statements, stemming from Lemma 3.1, complete our pursuit of consistency under convex-type multivariate stochastic orderings.

Proposition 3.1. *Assume $\mathbb{R}_+^d \subseteq K$. Then, $R_{\mathcal{H}}(\cdot)$ is $\leq_{\mathcal{F}^{d,d}_{\mathrm{decx}}}$-consistent: for any $U, V \in L_d^\infty(\Omega)$,*

$$U \leq_{\mathcal{F}^{d,d}_{\mathrm{decx}}} V \quad \Longrightarrow \quad R_{\mathcal{H}}(V) \subseteq R_{\mathcal{H}}(U).$$

Proof. Let $U, V \in L_d^\infty(\Omega)$ satisfy $U \leq_{\mathcal{F}^{d,d}_{\mathrm{decx}}} V$ (according to Definition 2.1.**i**, and (34)). Then, invoking a Strassen's representation theorem, as referenced in [33] and employed throughout the proof of Theorem 3.10 in [5], there exist versions $\tilde{U} \sim U$ and $\tilde{V} \sim V$ on $(\Omega, \mathcal{F}, \mathbf{P})$ such that, $\mathbf{P}$-a.s.,

$$\mathbf{E}\big[\tilde{V} \big| \tilde{U}\big] \leq_{\mathbb{R}_+^d} \tilde{U}.$$

Since $\mathbb{R}_+^d \subseteq K$, it follows that $\mathbf{E}\big[\tilde{V} \,\big|\, \tilde{U}\,\big] \leq_K \tilde{U}$, and consequently, by (48), Lemma 3.1, and also **(R-mon)** for $R_{\mathcal{H}}(\cdot)$, as in Subsection 1.1 of Section 1, we reach

$$R_{\mathcal{H}}(V) = R_{\mathcal{H}}\big(\tilde{V}\big) \subseteq R_{\mathcal{H}}\Big(\mathbf{E}\big[\tilde{V} \big| \tilde{U}\big]\Big) \subseteq R_{\mathcal{H}}\big(\tilde{U}\big) = R_{\mathcal{H}}(U). \qquad \square$$

Corollary 3.2. *Assume $\mathbb{R}_+^d \subseteq K$. Then, $R_{\mathcal{H}}(\cdot)$ is $\leq_{\mathcal{F}^{d,d}_{\mathrm{cx}}}$-consistent: for any $U, V \in L_d^\infty(\Omega)$,*

$$U \leq_{\mathcal{F}^{d,d}_{\mathrm{cx}}} V \quad \Longrightarrow \quad R_{\mathcal{H}}(V) \subseteq R_{\mathcal{H}}(U).$$

Proof. This emerges as a straightforward corollary of Proposition 3.1, since the evident set inclusion $\mathcal{F}^{d,d}_{\mathrm{decx}} \subseteq \mathcal{F}^{d,d}_{\mathrm{cx}}$ implies that

$$\leq_{\mathcal{F}^{d,d}_{\mathrm{cx}}} \quad \Longrightarrow \quad \leq_{\mathcal{F}^{d,d}_{\mathrm{decx}}}$$

(see Definition 2.1.**i**, and (34)). □

We end the current section, and thus the paper, by furnishing a case of remarkable law-invariant convex set-valued risk measures, as exemplified, for instance, in [9, Section 4].

Example 3.1. For any $i = 1, \ldots, M$, consider a concave *distortion function* $G_i \colon [0,1] \to [0,1]$, i.e., a non-decreasing function with $G_i(0) = 0$ and $G_i(1) = 1$. The *set-valued distortion risk measure* $R_G \colon L_d^\infty(\Omega) \to \mathcal{P}\big(\mathbb{R}^d\big)$, associated with $G \coloneqq [G_1, \ldots, G_m]^\mathsf{T} \colon [0,1] \to [0,1]^m$, is given by

$$R_G(U) \coloneqq \Big\{\big[G_i \circ \mathbf{P}(U_i)\big]_{i=1}^m + \mathbb{R}_+^m\Big\} \times \{0\}^{d-m}, \quad U \in L_d^\infty(\Omega)$$

where, for any $i = 1, \dots, M$, $G_i \circ \mathbf{P}(U_i)$ indicates the *Choquet integral* of U_i w.r.t. $G_i \circ \mathbf{P}$:

$$G_i \circ \mathbf{P}(U_i) = \int_0^\infty G_i\big(\mathbf{P}[U_i > x]\big)\, dx + \int_{-\infty}^0 \Big\{ G_i\big(\mathbf{P}[U_i > x]\big) - 1 \Big\} dx.$$

If, for any $i = 1, \dots, M$, G_i is continuous in 0, then $R_G(\cdot)$ is a *(set-valued) weighted value at risk* (see Theorem 4.1 in [9]): there exists $\mu \equiv \mu^G := [\mu_1, \dots, \mu_d]^\mathsf{T}$ where, for any $j = 1, \dots, d$, $\mu_j \equiv \mu_j^G$ is a probability measure on $\mathcal{B}\big(]0,1]\big)$, such that, for any $U \in L_d^\infty(\Omega)$,

$$\begin{aligned} R_G(U) &= \mathrm{WAV@R}_\mu(U) \\ &\doteq \left\{ \left[\int_{]0,1]} \frac{1}{\lambda} \mathbf{E}[V_j]\, \mu_j(d\lambda) - z_j \right]_{j=1}^d \,\middle|\, V \in L^1_{d,+}(\Omega),\, z \in \mathbb{R}^d,\, U + V - z \geq 0 \right\} \bigcap M \\ &= \left\{ \left[\inf_{z \in \mathbb{R}} \int_{]0,1]} \frac{1}{\lambda} \mathbf{E}\Big[(z - U_i)^+\Big] \mu_i(d\lambda) - z \right]_{i=1}^m + \mathbb{R}^m_+ \right\} \times \{0\}^{d-m}. \end{aligned}$$

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